Chapter 6

An Application of Network Synthesis to the Analysis of Interactive Behavior

In this chapter the concepts of a passive physical equivalent and an uncontrollable element are introduced. These concepts are intended to exploit physical systems modeling techniques to lend insight into the interactive behavior of controlled systems. In the next chapter passive physical equivalents are used to analyze the behavior of a force feedback controlled manipulator. In both this chapter and the next, attention will be restricted to 1-port systems.

6.1 Passive Physical Equivalents

The previous chapters have established that the properties of the driving point impedance dictate interactive behavior. For instance, the driving point impedance contains the only information necessary to predict the coupled stability properties of a linear manipulator. The impedance, however, is limited in one important respect: it can be an unwieldy tool for design.

A comparison with SISO servo design will illustrate this limitation. Consider the
Nyquist plot of an open loop transfer function $K(s)G(s)$, where $K(s)$ represents the compensator, and $G(s)$ the plant. The phase and gain margins which are taken from the Nyquist plot can be used directly for design. For instance, because the phase of the compensator adds to that of the plant, a change in the phase margin can be implemented simply by changing the phase of the compensator by an equal amount. The effect of changes in the plant dynamics on overall system behavior are equally apparent.

The Nyquist plot of the driving point impedance, however, despite its importance as an analytical tool, does not let effects due to the compensator and plant be separated in such a simple fashion. For instance, consider a manipulator with an open loop admittance $Y(s)$ relating the input effort, $e$, to the output flow, $f$, and an open loop transfer function $G(s)$ relating the control input, $u$, to $f$. If the control law $u = -K(s)f$ is implemented, then it is readily shown that the closed loop admittance is:

$$Y_c(s) = (I + G(s)K(s))^{-1}Y(s)$$

The role of $K(s)$ is now clouded because it does not show up as a simple multiplier. Whereas the concepts of gain and phase margin can be used to restrict the class of compensators which will lead to closed loop stability, there is no apparent method to restrict the class of compensators which lead to a positive real driving point impedance. The problem is only exacerbated if the sensed outputs are not the same as the interaction port outputs, or if input feedback is used.

What is needed is a way to simplify the interpretation of a closed loop driving point impedance. A successful method would clarify the contributions of the compensator and the plant to the overall behavior, perhaps going so far as to suggest changes to the design of either that would improve the behavior. The proposed method, the generation of a passive physical equivalent, has, in theory, this as well as other advantages to recommend it.
As indicated in Section 2.4.2, if a driving point impedance is positive real, then some passive system (called a realization) exists which has that impedance at an interaction port. If the driving point impedance is that of some controlled system, such a passive realization is termed a passive physical equivalent. In other words, although the system is feedback controlled, and therefore active at some sufficiently detailed level of description, there exists a passive system\(^1\) which has the identical port behavior. Passive physical equivalents may be found from driving point impedances using the network synthesis techniques described in the next section.

It should be evident that passive physical equivalents are a form of Hogan's "physical equivalence" described in Section 1.2.2. The two stated goals of the physical equivalence approach are one, to bring to bear on the behavior of a controlled system those powerful techniques developed for the analysis of physical systems, and two, to restrict in some useful fashion the class of controllers that need be considered. Passive physical equivalents will to a certain extent, given that stable interaction with arbitrary passive environments is a primary consideration, satisfy both goals.

For instance, consider Tellegen's theorem, which is one of the more powerful tools of physical systems theory. One consequence of Tellegen's theorem is that, if a system is composed of passive components, it is itself passive [4]. Although earlier we assumed that the driving point impedance was positive real, we evidently need not have—if we can show that each element of the passive physical equivalent is passive (i.e. positive viscous coefficients, stiffnesses, and masses), then we are certain that the overall system is passive\(^2\). This is an interesting result, because it is potentially much easier to determine whether a given element value, say a stiffness, is positive than it is to check the positive realness of a high-order impedance function. Furthermore, the dependence of an element value on control gains and plant parameters can be exploited either to

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\(^1\) Generally, multiple passive physical equivalents exist. See the next section.

\(^2\) The converse (an active element implying an active overall system) is not necessarily true, however. See Section 6.3.
bound gains or to direct tradeoffs between plant design and compensator design.

Finally, it is felt that the most important reason for using passive physical equivalents is to reap the insight that they can provide into the workings of control systems. For instance, it may be seen that the effect of a particular controller is to increase some stiffness or to decrease a mass. These are effects for which great insight exists—based, of course, on physical systems modeling. This capability and the others described above are used in the analysis of force feedback control presented in the next chapter.

Before introducing the network synthesis techniques, a simple example should help illustrate the passive physical equivalent concept. Consider the system shown in Figure 6.1(a), which has the following driving point admittance:

\[ Y(s) = \frac{s}{Ms^2 + K} \]

If the control law \( u = -kx - bv \) is implemented, the closed loop driving point admittance is:

\[ Y_c(s) = \frac{s}{Ms^2 + bs + (k + K)} \]

Because this controller is simply generating a force proportional to displacement and a force proportional to velocity, it is doing the equivalent of connecting the mass to ground through a spring of stiffness \( k \) and a damper of viscous coefficient \( b \). The two springs \( (k \) and \( K \) are in mechanical parallel, so their stiffnesses add. The passive physical equivalent is shown in Figure 6.1(b). It is readily shown that this system has the driving point admittance \( Y_c(s) \). For the equivalent to be passive, it is sufficient (according to Tellegen’s theorem) that \( b \geq 0 \) and \( k \geq K \); it is fairly easy to convince oneself that these conditions are also necessary for passivity.

This example points out several basic, but interesting facts:

- Control does not necessarily affect all elements of the plant model. In this case, the endpoint mass is unchanged.
Figure 6.1: An example of a passive physical equivalent. (a) The open loop system. \( u \) is the control force, and \( F \) is the force imposed by the environment. \( x \) and \( v \) are position and velocity states, measured relative to ground. (b) The passive physical equivalent, given the control law \( u = -kx - bv \) and \( b \geq 0 \) and \( k \geq -K \).

- Control can introduce elements, such as the damper.
- Control can affect the magnitude of elements, such as the spring.
- The coupled stability criterion does not always require that the control be dissipative. In this case, negative stiffness can be added. If inherent damping existed, negative damping could also be added.

In the next section, the network synthesis procedures which allow the generation of more complicated passive physical equivalents will be introduced.

6.2 Network Synthesis Procedures

In the examples of the last section, the result was correctly guessed by correlating the effect of the control to the behavior of physical elements. In general, this is both difficult and unnecessary. Several techniques can be found in the electrical engineering literature for the synthesis of two-terminal (1-port) networks from driving point impedance functions. There are several excellent texts which describe these techniques, including those by Van Valkenburg [85] and Weinberg [87].
The oldest and most widely used procedures apply to electric networks which contain two types of elements only, i.e., LC, RC, or RL networks. We will consider LC networks, as the techniques are easily generalized to the others. The first two methods are due to Foster, and date to 1924. Both are based on a partial fraction expansion of the driving point impedance. The first uses the impedance form, and the second the admittance form. The partial fraction expansions, along with circuit diagrams and bond graph realizations, are shown in Figures 6.2 and 6.3.

The Foster synthesis techniques, and, for that matter all the techniques to be considered here, proceed in cycles. The first cycle consists of extracting a pole at \( s = \infty \), if one exists. This is done by finding the residue, \( \alpha_1 \), of \( Y(s) / s \) as \( s \to \infty \). The second cycle consists of extracting a pole at \( s = 0 \), if it exists; this leads to the term \( \alpha_2 / s \). All subsequent Foster cycles consist of identifying sets of imaginary poles, and extracting them by finding their residues. The result has one of the two forms shown in Figures 6.2 and 6.3.

The second two methods for the synthesis of LC networks are due to Cauer, and date to 1927. These are: "continued fraction expansion about infinity" (Figure 6.4), and "continued fraction expansion about zero" (Figure 6.5).

Continued fraction expansion about infinity consists of the following steps: \( Z(s) \) is written in acausal form so that there is a pole at \( s = \infty \). This pole is removed by long division, leaving \( Z_1(s) \). \( Z_1(s) \) is inverted, yielding \( Y_1(s) \) which has a pole at \( s = \infty \). Each subsequent cycle consists of an inversion and the removal of a pole at \( s = \infty \). As indicated in Figure 6.4, the result of this process is a chain of inductors and capacitors, with the inductors in series, and the capacitors in parallel.

Continued fraction expansion about zero is similar, except that each cycle consists of an inversion and the removal of a pole at \( s = 0 \). The resulting network has the same structure, but with the capacitors in series and the inductors in parallel.
\[ Z(s) = \alpha_1 s + \frac{\alpha_2}{s} + \sum_{i=3}^{i=j-1} \frac{2\alpha_i s}{s^2 + \omega_i^2} + Z_j(s) \]

Figure 6.2: First Foster form—partial fraction expansion in the impedance form. The components of this expansion are all connected in series.
\[ Y(s) = \alpha_1 s + \frac{\alpha_2}{s} + \sum_{i=3}^{i=j-1} \frac{2\alpha_i s}{s^2 + \omega_i^2} + Y_j(s) \]

Figure 6.3: Second Foster form—partial fraction expansion in the admittance form. The components of this expansion are all connected in parallel.
Figure 6.4: First Cauer form—continued fraction expansion about infinity. When applied to the impedance of a positive real LC network, this synthesis technique results in a chain of series inductors and parallel capacitors.
\[ Z(s) = \frac{1}{C_1 s} + \frac{1}{L_2 s} \left( \frac{1}{C_3 s} + \frac{1}{L_4 s} + \cdots \right) \]

Figure 6.5: Second Cauer form—continued fraction expansion about zero. When applied to the impedance of a positive real LC network, this synthesis technique results in a chain of series capacitors and parallel inductors.
The synthesis of RC and RL networks is essentially identical. Either the Foster or the Cauer techniques can be applied to these networks with little difficulty. One can even find a change of variables which will convert an RC network to an LC network or vice versa, or an RL to an LC or vice versa [87]. One difference, however, is worth noting. Whereas the real part of the driving point impedance of an LC network is zero for all \( s = j\omega \), it is greater than zero for an RC or RL network. In these cases, however, the real part is a monotonic function of \( \omega \) which has its minimum at either 0 or \( \infty \). This minimum, if not zero, is generally removed as a first step, corresponding to the removal of some series or parallel resistance.

An example of the generation of a passive physical equivalent using Cauer synthesis is presented in Appendix C.

The Foster and Cauer syntheses owe much of their simplicity to the fact that, for networks composed of two kinds of elements only, after any cycle, the remaining impedance is guaranteed to be positive real [85]. For RLC circuits, however, this need not be the case. At some point in the synthesis process, it is possible that the remaining impedance will be a minimum function, i.e. an impedance whose real part is not a monotonic function of \( \omega \), but has a minimum value at (at least one) finite frequency, \( \omega_1 \). If an appropriate resistor is removed so that this minimum value becomes zero, it can be shown that no poles or zeros remain on the imaginary or real axes, and that none of the previous techniques can be successfully applied for another cycle without a remainder function which is not positive real. As a result, a new type of cycle is needed.

Such a cycle was provided by Brune in 1931 [13]. The Brune cycle entails the removal of the circuit element shown in Figure 6.6(a). This element consists of an ideal\(^3\) transformer (unity coefficient of coupling: \( M/\sqrt{L_p L_s} = 1 \)) connected to a capacitor.

\(^3\)"Ideal" in the electrical engineering sense of no flux leakage rather than the bond graph sense of no energy storage.
Brune showed that when such an element, with \( L_p, L_s, M, \) and \( C \) properly chosen, is removed from a minimum function, the result is positive real, and both numerator and denominator are reduced in order by two.

A bond graph of this circuit element is shown in Figure 6.6(b). As the bond graph indicates, energy is stored but not dissipated in an ideal transformer. Energy is stored in a magnetic field (in the soft iron loop connecting the coils) with the constitutive relation \( \dot{\mathcal{F}} = \mathcal{R} \dot{\Phi} \) (see figure legend) which is considered to be that of a capacitor. Energy is transformed\(^4\) from the magnetic domain to the electrical when changes in the magnetic flux, \( \Phi \), induce electromotive force in the coils. Of course, the reverse process also occurs, i.e., current in the coils induces a magnetomotive force.

Given this understanding, it is fairly straightforward to find a mechanical equivalent. Gyrators, however, do not exist in the translational mechanical realm. Transformers, therefore, are used instead, as shown in Figure 6.7(a). Ignoring for a moment the storage, we note that two gyrators connected in series constitute a transformer, as do two transformers connected in series. Both descriptions, therefore, have the basic behavior of a transformer.

The mechanical equivalent shown in Figure 6.7(b) is somewhat awkward. A simpler equivalent can be found by generating its dual, which is shown in Figure 6.8. In this case, the ideal transformer is simply a massless lever; the storage in the transformer is the rotational spring at its pivot. The capacitor in the Brune element becomes a mass upon which the lever assembly rests.

Although the dual is a somewhat simpler equivalent, it is unfortunate to resort to ideal transformers at all, as they surely add a level of complexity to the interpretation of a passive physical equivalent. Ideal transformers are a source of concern for those interested in network synthesis also, primarily for reasons of parameter sensitivity which are discussed in the next section. After Brune published his method in 1931,

\(^4\)Given \( \mathcal{F} \) as efforts, the relations \( \mathcal{F} = N \dot{\Phi} \) and \( \mathcal{F} = N i \) are actually those of a gyrator.
Figure 6.6: The Brune element. (a) Circuit diagram of the circuit element extracted by one Brune cycle. (b) Bond graph equivalent to the circuit diagram. $v$ is voltage, $i$ is current, $N$ is number of turns, $\Phi$ is the rate of change of magnetic flux, $\mathcal{F}$ is magnetomotive force, and $\mathcal{R}$ is reluctance. The constitutive equation of the magnetic storage is $\dot{\mathcal{F}} = \mathcal{R} \dot{\Phi}$, and of the gyrators are $v = N \dot{\Phi}$ and $\mathcal{F} = Ni$. The bond graph model leads to the same equations as the circuit model provided we identify the inductances as follows: $L_p \equiv N_1^2 / \mathcal{R}$, $L_s \equiv N_2^2 / \mathcal{R}$, and $M \equiv N_1 N_2 / \mathcal{R}$. 

\[ v_1 - v_0 = L_p \frac{di_p}{dt} + M \frac{di_s}{dt} \]
\[ v_2 - v_0 = M \frac{di_p}{dt} + L_s \frac{di_s}{dt} \]
\[ i_p + i_s = C \frac{dv_0}{dt} \]
Figure 6.7: Mechanical equivalent to Brune element. (a) Bond graph model. (b) Mechanical equivalent. Definitions and assumptions: \( r = l_i/l_o \); no gravity; cables are inextensible and never go slack; levers and pulleys are massless; \( f_0 \) increases in compression.
a tremendous amount of effort was put into the search for "transformerless" synthesis methods.

This search bore no fruit until 1949 when Bott and Duffin [11] showed that such a method did indeed exist. This method was an important milestone in the history of network synthesis. It is explained in detail in [85,87]. We will not review it here, however, because it results in a realization with many more elements than that generated by the Brune synthesis. The Brune realization is minimal, meaning that no realization can be found with fewer elements, whereas the Bott-Duffin realization is non-minimal, meaning that it results in a higher-order impedance which relies upon pole-zero cancellations to generate the correct driving point behavior. Thus, the Bott-Duffin synthesis has two serious drawbacks for our purposes: it generates a more complex realization measured in terms of the number of elements, and it relies upon exact pole-zero cancellation, which is a phenomenon we associate with control systems, usually not with physical systems. It is therefore inconsistent with one of our primary reasons for generating a passive physical equivalent, to generate insight through the application of physical systems modeling techniques.

This completes our brief tour of network synthesis procedures. In summary, given a positive real driving point impedance, a passive physical equivalent can always be found, but there is no guarantee that it will be simple to interpret. As for electrical network synthesis, the Bott-Duffin and all subsequent synthesis techniques turned out to exhibit extreme parameter sensitivity and to be of little value in practice. In the 1980's, analytical network synthesis procedures are rarely ever taught or used. The next section describes how parameter sensitivity and various other difficulties affect passive physical equivalents.
Figure 6.8: Dual of mechanical equivalent. (a) Bond graph model. (b) Dual. Definitions and assumptions: no gravity; cable is inextensible and never goes slack; lever and pulley are massless.
6.3 Difficulties

After the previous section, it should be evident that there are several potential difficulties with the generation and interpretation of passive physical equivalents. The various difficulties as they are currently perceived are discussed in this section. They are:

- Complexity introduced by ideal transformers.
- Parameter sensitivity of realizations.
- Non-uniqueness of realizations.
- Structural stability of realizations.
- Complexity of an analytical approach.

Upon close inspection, the complexity introduced by the ideal transformers that appear in the Brune synthesis is not really a problem at all. Despite the complicated junction structure created by the lever arrangement in, for instance, Figure 6.7, the passivity of the element is determined only by the signs of the stiffness, $K$, and the mass, $M$, exactly as it is for any other second order lossless mechanical element. The junction structure is simply a price that must be paid to understand the behavior of a sufficiently complicated control system.

It is more important to realize that the parameter sensitivity of the Brune (and other) syntheses is not a difficulty, either. The problem with electrical network synthesis is that the realizations are meant to be built. However, if the coefficient of coupling of the transformer is not exactly unity (or the pole-zero cancellations of the Bott-Duffin synthesis are not exact), the circuits may end up exhibiting behavior which differs substantially from that of the desired impedance. But the passive physical equivalent is not meant to be built—it has already been implemented by the control system. We need be no more concerned with an ideal transformer than with a massless spring.
And, although we do not recommend the use of the Bott-Duffin technique, any pole-zero cancellation would be exact by definition, as the order of the closed loop admittance is fixed.

The biggest problem with passive physical equivalents is probably one of the most useful features for the network synthesist. This is the non-uniqueness of the realizations. Consider, for instance, I.C networks. Four different synthesis techniques are available, and there is nothing that prevents the mixed use of these techniques. Thus, for a network of high enough order, the number of valid but distinct realizations may be quite large.

This is not necessarily a problem if a single realization can be selected for analysis. Of course, there is the problem of selecting the most appropriate realization, but one can generally use a model of the open loop system as a guide by attempting to select the realization with the structure closest to this model. This is the approach that is taken in Chapter 7. It tends to work well for low-order compensators and controllers (e.g. impedance controllers) which are intended to exhibit physically-interpretable behavior, but not as well for high-order compensators which are designed to meet certain performance criteria.

A more serious problem, however, is that, although there is, by definition, a one to one mapping from the driving point impedance of each realization to that of the closed loop system, there is not necessarily a one to one mapping from the element by element passivity of a given realization to the passivity of the closed loop system.

For instance, suppose that the controller includes some gain factor, $G$, and that we know from an analysis of the driving point impedance that the positive real condition is satisfied so long as $a \leq G \leq c$. Then it does not follow that any realization will contain all positive element values for the same range of $G$. What does follow is that some set of realizations must exist which collectively contain instances of all positive element
values for all \( a \leq G \leq c^5 \). For instance, one realization may be valid for \( a \leq G \leq b \), but at \( G = b \) the viscous coefficient of some damper goes to 0, and is negative for \( G > b \). Meanwhile, another realization, which had a negative damper in some other location for \( G < b \), has this damper become positive for \( b \leq G \leq c \). An interesting consequence is that changing feedback gains can change the structure of the passive physical equivalent.

Another important consequence is that negative element values do not necessarily indicate a violation of the positive real condition. For this, it is necessary that every possible minimal realization have at least one negative element. By "minimal" we mean the equivalent of "reduced" in the bond graph sense, so that no excess storages or dissipators exist. If not reduced, it is possible that a negative damper, for instance, could be added to a larger positive damper, thus removing the negative element value.

To summarize:

- If all element values of a realization are positive, the driving point impedance is positive real.

- If a minimal realization contains a negative element value, the driving point impedance violates the positive real condition only if all other minimal realizations derived from that impedance also contain negative elements.

The fourth area of concern is the "structural stability" of the realizations. We have already indicated that, as a gain changes, the structure of the realization may change. Presumably, the same thing could happen due to a change in a plant parameter. If the nominal value of the plant parameter was close to its critical value, modeling uncertainties would introduce uncertainty with regard to the structure of the passive physical equivalent. Of course, so long as the modeling uncertainties introduce only

\(^5\)Of course, no realization can exhibit all positive element values outside this range.
small errors to the driving point impedance, the two structures would necessarily exhibit similar port behavior.

A reasonable approach to treating this problem would be to develop software for the generation of passive physical equivalents. A software package similar to those which have been developed for the analysis of electrical networks and physical systems would graphically display the parameter and structural changes of the passive physical equivalent resulting from changes in plant or compensator parameters. The software might also address the uniqueness problem by sorting among the realizations to find those with the "best" structures (based, perhaps, on similarity to the plant structure).

Software is also the logical approach to the fifth area of concern, the complexity of an analytical generation of passive physical equivalents. All passive physical equivalents in this thesis were determined symbolically in terms of plant and compensator parameters. Even though only simple systems were treated, a great deal of time and effort was involved. In conclusion, it is felt that the future of passive physical equivalents as a technique for the analysis of interactive behavior depends in large part on the development of appropriate software. Certainly the Ricatti equation would never have developed such fame without the help of digital computers.

6.4 Uncontrollable Elements

Although the generation of a passive physical equivalent may in some cases be prohibitively difficult, a complete realization is not always necessary to learn much that is useful. For instance, it may be valuable to know, given particular actuator and sensor locations, which elements of the plant model cannot be changed by any compensator; i.e., which elements must appear, unaltered, in any passive physical equivalent, given any (causal) compensator. Such "uncontrollable elements" are the subject of this
section.

It can be shown that, given a plant model sufficiently detailed to include actuator and/or transmission dynamics separating the control and interaction ports, that at least one uncontrollable element must exist. The reasoning is as follows.

In Appendix D it is shown that, given any causal compensator and non-colocated control and interaction ports, the relative order of the driving point impedance will be unchanged and furthermore, that:

$$\lim_{s \to \infty} Z(s) = \lim_{s \to \infty} Z_c(s)$$

where $Z_c(s)$ is the closed loop driving point impedance. If $Z(s)$ is strictly proper, this implies that the zero at $s = \infty$ is unaffected by compensation. This in turn implies that any of the possible realizations contains some element which is unaffected by compensation. This is because all of the synthesis techniques result in a term of the form $a/s$, where $a = \lim_{s \to \infty} s Z(s)$, if a zero exists at $s = \infty$.

The preservation of the limit as $s \to \infty$ may be understood physically to mean that the high frequency behavior, which is dictated by the element located at the interaction port\(^6\), cannot be altered by a controller which is necessarily of limited bandwidth. Therefore, that element located at the interaction port is preserved in any passive physical equivalent. Such an element is termed "uncontrollable" because no causal compensation scheme can alter its value.

Barring colocated control and interaction ports and barring actuators and transmissions of infinite bandwidth, every control system will contain an uncontrollable element. However, the element found at the interaction port is not necessarily the only one which is uncontrollable. Given a particular configuration of actuators and sensors, there may be others. Examples of multiple uncontrollable elements will play an important role in the analyses of the next chapter.

\(^{6}\)If the plant is an admittance, this element must be a mass, if the plant is an impedance, it must be a spring.
The concept of an uncontrollable element illustrates a very important idea, which is that there is at least one aspect of the behavior of a physical system which no controller, no matter how sophisticated, can alter. Furthermore, this aspect is not one of small consequence, rather it is a critical determinant of the behavior which the control system represents to the outside world.