Chapter 4

Examples of the Coupled Stability Criterion

The purpose of this chapter is to provide a number of examples that illustrate the coupled stability criterion and its application to the analysis of closed loop designs. The examples will focus on the consequences of failure to satisfy the criterion. A number of the examples are of controllers designed for a two-link manipulator. These controllers were implemented on an actual manipulator, and interactive behavior was experimentally examined. The experiments and results are described in the next chapter.

4.1 Impedance Controllers

This section examines the coupled stability properties of various impedance control implementations that have been described in the literature. Of special interest is the linearized control presented in Section 4.1.2, because an experimental investigation of its properties is described in the next chapter.
4.1.1 Feedback Linearized Impedance Control

In [35], Hogan presents an impedance controller for a manipulator with the following rigid body description:

\[ I(\theta) \frac{d\omega}{dt} + C(\theta, \omega) + V(\omega) + S(\theta) = U + J'(\theta)F \]

where \( \theta \) and \( \omega \) are the generalized joint angles and velocities, respectively; \( I(\theta) \) is the configuration-dependent inertia tensor; \( C(\theta, \omega) \) is a vector of inertial coupling terms; \( V(\omega) \) includes any velocity-dependent forces such as friction; \( S(\theta) \) includes any static configuration-dependent forces such as gravitational forces; \( U \) is a vector of actuator torques; \( J(\theta) \) is the configuration-dependent jacobian relating joint velocities to endpoint velocities; and \( F \) is a vector of endpoint forces imposed by the environment (also to be called interaction port forces).

The desired endpoint behavior of the manipulator is given by:

\[ M \frac{dV}{dt} - B[V_0 - V] - K[X_0 - X] = F \]

where \( X \) and \( V \) represent the position and velocity of the endpoint, and \( X_0 \) and \( V_0 \) represent the commanded position and velocity. This behavior is clearly that of a passive, linear system for PD \( M \), \( B \), and \( K \).

Simply by equating the endpoint accelerations of the target and actual models, a control (expression for \( U \) in terms of \( \theta \), \( \omega \), and \( F \)) can be derived which will generate the target behavior exactly! Thus, even though this manipulator is nonlinear, it should be possible to satisfy the coupled stability criterion.

4.1.2 A Linearized Impedance Controller for a Two-Link Manipulator

The geometry of the two-link manipulator used in the experiments is shown in Figure 4.1. Nonlinear equations describing the dynamic behavior of this manipulator are
derived in [23]. They are:

\[ I(\theta)\ddot{\theta} = \beta \omega + C(\theta, \omega) + U + J'(\theta)F \]

where:

\[
I(\theta) = \begin{bmatrix}
J_1 + l_1^2 m_2 & l_1 h_2 m_2 \cos(\theta_2 - \theta_1) \\
l_1 h_2 m_2 \cos(\theta_2 - \theta_1) & J_2
\end{bmatrix}
\]

\[
\beta = \begin{bmatrix}
-(B_1 + B_c) & B_c \\
B_c & -(B_2 + B_c)
\end{bmatrix}
\]

\[
C(\theta, \omega) = \begin{bmatrix}
l_1 h_2 m_2 \omega_2^2 \sin(\theta_2 - \theta_1) \\
l_1 h_2 m_2 \omega_2^2 \sin(\theta_2 - \theta_1)
\end{bmatrix}
\]

\[
J(\theta) = \begin{bmatrix}
-l_1 \sin \theta_1 & -l_2 \sin \theta_2 \\
l_1 \cos \theta_1 & l_2 \cos \theta_2
\end{bmatrix}
\]

\[ \theta_1 = \text{absolute angle of link 1 (rad)} \]
\[ \theta_2 = \text{absolute angle of link 2 (rad)} \]
\[ \omega_1 = \text{angular velocity of link 1 (rad/sec)} \]
\[ \omega_2 = \text{angular velocity of link 2 (rad/sec)} \]
\[ l_1 = \text{length of link 1 (m)} \]
\[ l_2 = \text{length of link 2 (m)} \]
\[ h_2 = \text{length from pivot to c.g. of link 2 (m)} \]
\[ m_2 = \text{mass of link 2 (kg)} \]
\[ J_1 = \text{moment of inertia of link 1 (about joint) (kg-m}^2) \]
\[ J_2 = \text{moment of inertia of link 2 (about joint) (kg-m}^2) \]
\[ B_1 = \text{viscous damping, link 1 to ground (kg-m}^2/\text{s}) \]
\[ B_2 = \text{viscous damping, link 2 to ground (kg-m}^2/\text{s}) \]
\[ B_1 = \text{viscous damping between links (kg-m}^2/\text{s}) \]

**Linearization**

Because the experiments described in the next chapter were performed only in the vicinity of certain zero-velocity operating points, these equations of motion were linearized about those operating points. The reader may question the need for this if the
impedance control law described in the last section could be used to linearize the endpoint behavior. That control law, however, is computationally expensive, and was not judged to be necessary for the small motion experiments. The linearized manipulator description is:

$$I(\Delta \theta_0)\dddot{\theta} = \beta \dot{\theta} + U + J'(\theta_0)F$$

where $\delta \theta = \theta - \theta_0$, $\omega_0 = 0$, and $\Delta \theta_0 = \theta_2 - \theta_1$. Dropping the $\delta$, this can be written in standard state variable form as follows:

$$
\begin{bmatrix}
\dot{\theta} \\
\dot{\omega}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & I^{-1}\beta
\end{bmatrix}
\begin{bmatrix}
\theta \\
\omega
\end{bmatrix} +
\begin{bmatrix}
0 \\
I^{-1}
\end{bmatrix}
U +
\begin{bmatrix}
0 \\
I^{-1}J'
\end{bmatrix}
F
$$

or:

$$\dot{\Theta} = A\Theta + B U + L F$$

(Control Derivation)

For what will be called the "simple impedance controller" to be designed and implemented here, no attempt will be made to measure and use force information; only joint
angles and velocities will be used. The derivation of this control law follows that given by Hogan in [35]. The dynamics of the manipulator are ignored, and the principle of virtual work is used to relate the actuator torques to the desired endpoint force with the following result:

\[ U = J'F \]

The desired endpoint behavior is given by:

\[ F = -K_2X - B_2V \]

where the commanded position and velocity are zero. Then, for small displacements:

\[ U = -J'K_2J\theta - J'B_2J\omega \]

It is a simple matter to provide additional compensation for the viscous losses in the manipulator:

\[ U = -J'K_2J\theta - (J'B_2J + \beta)\omega \]

or,

\[ U = -G\Theta \]

\[ G = \begin{bmatrix} J'K_2J & J'B_2J + \beta \end{bmatrix} \]

The closed loop state equations are then:

\[ \dot{\Theta} = (A - BG)\Theta + LF \]

\[ X = \begin{bmatrix} J & 0 \end{bmatrix}\Theta \]

Figure 4.2 is a Nyquist plot of the closed loop driving point admittance\(^1\) of this system for typical values. The closed loop system clearly appears passive.

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\(^1\) Actually, because the manipulator has two degrees of freedom, an n-port passivity criterion should be used. This will be done in the next chapter; for the present, however, only the diagonal terms will be considered. These can be analyzed with the simpler 1-port techniques, and are representative (for the examples considered here) of all interesting features.
\[ \theta_{10} = 30^\circ, \quad \theta_{20} = 150^\circ \]

\[
I = \begin{bmatrix}
0.1954 & -0.0560 \\
-0.0560 & 0.0932 \\
\end{bmatrix}, \quad \beta = \begin{bmatrix}
-0.01070 & 0.01066 \\
0.01066 & -0.01070 \\
\end{bmatrix}
\]

\[
J = \begin{bmatrix}
-0.183 & -0.168 \\
0.316 & -0.291 \\
\end{bmatrix}
\]

\[
K_z = \begin{bmatrix}
60 & 0 \\
0 & 60 \\
\end{bmatrix}, \quad B_z = \begin{bmatrix}
10 & 0 \\
0 & 10 \\
\end{bmatrix}
\]

Figure 4.2: Simple impedance controller. Nyquist plot of \( Y_{zz}(s) \). Both axes have units of sec/kg.
4.1.3 The Addition of First-Order Decoupled Actuator Dynamics

Kazerooni [44] developed a methodology for the design of impedance controllers for plants of the following form:

\[
\begin{bmatrix}
\Delta \dot{\Theta} \\
\Delta \ddot{\Theta} \\
\Delta \dot{T}
\end{bmatrix} =
\begin{bmatrix}
0 & I & 0 \\
-M^{-1}(\Theta_0)GR(\Theta_0) & M^{-1}(\Theta_0)T_r & 0 \\
0 & 0 & A_a
\end{bmatrix}
\begin{bmatrix}
\Delta \Theta \\
\Delta \dot{\Theta} \\
\Delta T
\end{bmatrix}
\]

\[+ \begin{bmatrix}
0 \\
0 \\
B_a
\end{bmatrix} \Delta U + \begin{bmatrix}
0 \\
M^{-1}(\Theta_0)J_c' \\
\end{bmatrix} \Delta D\]

Or, in short:

\[
\Delta \dot{X} = A \Delta X + B \Delta U + L \Delta D
\]

where all of the submatrices of $A$ are $n \times n$ (n d.o.f.); $\Theta_0$ ($\dot{\Theta}_0 = 0$) is the operating point of the system, about which the linearization occurs; $M$ is the inertia matrix of the system; $GR$ is a gravity term; $T_r$ is a nonsingular square matrix which represents the effect of the actuator torques, $\Delta T$, on the coordinates, $\Delta \Theta$; $A_a$ is a diagonal matrix of actuator bandwidths; $B_a = -A_a$; $\Delta U$ is the input to the actuators; $\Delta D$ is the force applied by the environment; and $J_c$ is a jacobian matrix relating the coordinates of the interaction ports to $\Delta \Theta$.

This form is quite similar to that shown in the last section, except that gravity is included, joint damping is omitted, and, most importantly, the actuators no longer behave as pure sources. Instead, each actuator has a first order rolloff described by one of the diagonal terms in the matrix $A_a$. The actuator dynamics remain decoupled, however, in the sense that there are no back effects from the dynamics of the hardware.

Kazerooni's method involves matching the behavior of this system to that of the following set of target dynamics, over some frequency range $0 < \omega < \omega_0$:

\[
\begin{bmatrix}
\Delta \dot{\Theta} \\
\Delta \ddot{\Theta}
\end{bmatrix} =
\begin{bmatrix}
0 & I \\
-J_c^{-1}J^{-1}KJ_c & -J_c^{-1}J^{-1}CJ_c
\end{bmatrix}
\begin{bmatrix}
\Delta \Theta \\
\Delta \dot{\Theta}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
J_c^{-1}J^{-1}
\end{bmatrix} \Delta D
\]

94
Figure 4.3: Closed loop control system, taken from [44]. \( G \) is a state feedback matrix, and \( G_d \) is an input feedback matrix.

where \( J \) is an inertia matrix, \( C \) is a damping matrix, and \( K \) is a stiffness matrix, all expressed in interaction port coordinates. The matrices \( J \), \( C \), and \( K \) are chosen by the designer to represent the desired dynamic response.

The details of this technique will not be reviewed here, as they are rather involved, but they include the use of an eigenvector assignment technique to design a state feedback matrix, as well as a novel technique for designing an input feedback matrix. The general form of the controller is shown in Figure 4.3.

Kazerooni provides a number of quantitative examples in [44]. One of these (example 1, case 1) applies to a two-link manipulator like that described in the previous section. In this example, the target dynamics are chosen to be soft in the \( x \)-direction at the endpoint, and about 20 times stiffer in the \( y \)-direction, as might be used to track a surface parallel to the \( y \)-axis.

The results may be summarized with Bode plots of the driving point admittance (Figure 4.4). The design specifically calls for matching the target behavior out to a bandwidth of 1 Hz (6.28 rad/sec), which is clearly achieved.

There is a problem, however. The phase of the impedance clearly dips below \(-90^\circ\),
Figure 4.4: Bode plots of closed loop driving point admittance (solid lines) and target model admittance (dashed lines). Example taken from example 1, case 1, [44]; upper left components of admittance matrices.
Figure 4.5: Nyquist plots of closed loop driving point admittance (solid lines) and target model admittance (dashed lines). Example taken from example 1, case 1, [44]; upper left components of admittance matrices.
and this appears in the Nyquist plot (Figure 4.5) as a loop in the left half plane. The consequence (as a worst environment test would reveal) is that this system is unstable upon interaction with stiff environments. This might not be so damaging, except that the goal of the procedure was to develop a controller suitable for contact applications.

Thus, even when explicitly included in a design procedure, first-order actuator dynamics complicate matters severely. In Chapter 8 it will be shown that the greatest source of difficulty is not the actuator dynamics alone, but rather the combination of input feedback and actuator dynamics.

### 4.2 Force Feedback

This section presents a simple analysis of a force-feedback controlled manipulator in an effort to explain the limitations of its interactive behavior. The control of force exerted on a rigid surface is certainly the example of manipulator interaction which has received the greatest attention in the literature. Part of the reason for this attention (besides the potential value of a force-controlled robot in a manufacturing setting) is the notorious "contact instability" associated with force-feedback. Contact instability is the phenomenon of violent chatter which can occur when a force-feedback controlled robot is brought into contact with a rigid surface. A detailed account of this phenomenon, along with a review of the relevant literature, is presented in Chapter 7.

For the present, an examination of the coupled stability properties of two simple models should elucidate one source of trouble. Consider, to begin, the manipulator model shown in Figure 4.6—a rigid body model reduced to its barest essence. If the force control law \( u = G(F_0 - F) \) is chosen for this manipulator, then the closed loop driving point admittance is:

\[
Y(s) = \frac{1 + G}{Ms}
\]
Figure 4.6: Rigid body manipulator. $u$ is the actuator force; $F$ is a force imposed by the environment.

Figure 4.7: Manipulator with non-colocated actuation and sensing.

This admittance will be positive real so long as $G \geq -1$. This is encouraging, because servo theory would suggest that the route to high-bandwidth force control would be to make $G$ as large as possible.

Now consider another simple manipulator model, shown in Figure 4.7. An essential feature of this model is that the actuator ($u$) and sensor ($F$) are non-colocated [25,20]. Non-colocation is given quantitative meaning in Appendix D; however, it can be understood as a "dynamic" separation of two physical locations—dynamic in the sense that power flowing between the locations must pass through energy storing elements. The reason for introducing non-colocation is described in Chapter 7; it is an essential factor in predicting contact instability.

The closed-loop admittance of this manipulator is:

$$Y(s) = \frac{s^2 + 2B((1 + G)/M)s + 2K((1 + G)/M)}{\frac{1}{2}Ms^3 + 2Bs^2 + 2Ks}$$

Passivity $6b$ can be used to check the positive realness of this admittance. Because $M > 0$, $B > 0$, and $K > 0$, $Y(s)$ has no poles in the RHP. It does, however, have
a pole at the origin. Although this pole is simple, its residue must be checked:

$$\lim_{s \to 0} sY(s) = \frac{1 + G}{M} \geq 0$$

Given that this is the residue associated with the rigid body mode, it should not be too surprising that it leads to the same result as the rigid body example above, namely, that $G \geq -1$.

In addition, however, it is necessary that $Re\{Y(j\omega)\} \geq 0$. It is straightforward to show that:

$$Re\{Y(j\omega)\} = \frac{(1 - G)B\omega^4}{M^2\omega^6 + 4(B^2 - MK)\omega^4 + 4K^2\omega^2}$$

The denominator of this expression is the sum of squares, so that the only new condition is $G \leq 1$.

The interesting result is that, no matter what the exact values of $M$, $B$, and $K$ are, so long as they are greater than zero, the coupled stability condition is $-1 \leq G \leq 1$.

Nyquist plots of this system are shown in Figure 4.8 for the following parameter values: $M = 2$, $B = 2$, and $K = 10,000$. The loops correspond to behavior near the structural resonance; the remainder of each Nyquist plot lies on the imaginary axis. The top plot is for $G = 0$, so that the system is passive, and as passivity requires, it lies completely in the closed right half plane. The lower plot is for the seemingly moderate force gain of $G = 2$, but the difference is rather dramatic. The loops now lie completely within the closed left half plane, and coupled stability is clearly in jeopardy.

The root locus test can be used to learn more about the types of environments which cause problems; this is shown in Figure 4.9. Evidently, it is springs which cause the difficulty. The worst instability occurs when the stiffness of the environment is comparable to the structural stiffness of the manipulator, however, any stiffer environment will also cause instability. To verify this, the centroid of the root locus can be found. The centroid is the point on the real axis through which the two asymptotes pass. Because the pole and zero locations in this example may be found analytically,
Figure 4.8: Nyquist plots of a force controlled manipulator's admittance. Open loop ($G = 0$), top; closed loop ($G = 2$), bottom.
Figure 4.9: Worst environment root loci for a force controlled manipulator.

it is straightforward to calculate the centroid location. The result is:

\[ \sigma_{\text{centroid}} = (G - 1) \frac{B}{M} \]

The conclusion is that, for \( G > 1 \), stiff enough environments must lead to instability.

A final note: springs are not the only environments which can lead to instability. A simple scaling of the Nyquist plot for \( G = 2 \) as would occur if the system were coupled to a large enough viscous damper, would be sufficient to create instability.
4.3 PID Control

This section examines the PID control of second order systems. A simple SISO control example is considered in some detail, and a digital implementation is developed for the two-link manipulator.

4.3.1 SISO Example

Consider the PID control of a second order system described by the following state equations ($M > 0; B, K \geq 0$):

\[
\begin{bmatrix}
\dot{x} \\
\dot{v}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-K/M & -B/M
\end{bmatrix} \begin{bmatrix}
x \\
v
\end{bmatrix} + \begin{bmatrix}
0 \\
1/M
\end{bmatrix} u + \begin{bmatrix}
0 \\
1/M
\end{bmatrix} F
\]

A block diagram representation of the closed loop system is shown in Figure 4.10, and the driving point admittance of this system is:

\[
Y(s) = \frac{s^2}{Ms^3 + (a + B)s^2 + (b + K)s + c}
\]

According to Corollary 3 of Section 2.4.1, the zeros of a positive real function must be simple, a condition which is violated by the double zero at the origin. A necessary condition for $Y(s)$ to be positive real is $c = 0$, in other words, the removal of all integral action. This is not very surprising in terms of conventional notions of passivity. The resulting admittance of the PD controlled system is:

\[
Y(s) = \frac{s}{Ms^2 + (a + B)s + (b + K)}
\]

It is relatively easy to show that the PD controller results in a positive real $Y(s)$ so long as $a \geq -B$ and $b \geq -K$. These are unrestricted conditions, as it is nearly always the case that $a$ and $b$ are greater than zero (corresponding to negative feedback and a minimum phase compensator zero). It should also be noted that the simple impedance
controller introduced previously is essentially a multivariable PD controller. Its robust coupled stability property should, therefore, be expected.

Returning to PID control, it is reasonable to ask what type of environment will lead to instability. Again, the root locus test may be used. Before proceeding with this, however, it would be useful to specify the compensator parameters more carefully.

Most techniques for selecting the three gains (a, b, and c) of a PID controller are based upon tradeoffs between certain performance specifications, such as time to peak, maximum overshoot, steady state error, and stability margin [17]. If the plant is second order, however, the gains may be selected to invert the plant. If we select \( a = M/\tau \), \( b = B/\tau \), and \( c = K/\tau \), then the zeros of the compensator exactly cancel the poles of the plant, leaving the well behaved loop transfer function \( 1/\tau s \). \( \tau \) is a free parameter which can be used to set the closed loop bandwidth. Given this choice of parameters, the driving point admittance is:

\[
Y(s) = \frac{s^2}{(s + 1/\tau)(Ms^2 + Bs + K)}
\]

The coupled stability properties of this admittance can be examined by generating the root loci for spring environments and mass environments. Examples of these are shown in Figure 4.11. Apparently, it is large masses that lead to difficulty. The minimum mass which will cause instability may be found analytically to be:

\[
(M_e)_{\min} = \frac{BM}{K\tau} + \frac{B^2}{K} + B\tau
\]
Figure 4.11: Second order system with a PID compensator: interaction with springs (left) and masses (right).

\[ \frac{B}{K_r} (M + B r + K r^2) \]

The dependence upon plant parameters indicates that increasing \( M \) and \( B \) increases the \( M_r \) necessary to create instability, whereas increasing \( K \) decreases the necessary \( M_r \); the dependence on \( r \) is not monotonic. In any case, a stiff, underdamped system will be driven unstable by a small mass. This may also be seen on the root locus; the closer the complex poles move toward the imaginary axis, the smaller the mass that is needed to make the locus cross the axis.

This analysis, of course, has its limitations, the most severe of which is that a pure derivative term \((as)\) cannot be implemented. In practice, the derivative action must be rolled off with at least one additional pole \((as/(T's + 1))\). This, of course, complicates the selection of \( r \); if one chooses \( T \) too small and \( r \) too large, the closed loop behavior may be unacceptably oscillatory.
4.3.2 Two-Link Manipulator Implementation

For the set of experiments involving PID control, the two-link manipulator was used as a test-bed for second order systems. To do this, the nominal elbow angle of the manipulator was set at 90° to decouple the links, and proportional position and velocity feedback loops were closed around each joint. The position and velocity feedback gains \((K\text{ and } B)\) could be used to set the pole locations for each link independently. The goal of the PID control would then be to cancel the pair of poles for each joint, and to insert instead the simple loop transfer function \(1/\tau s\).

This may appear to be a rather peculiar approach to manipulator control, adding dynamics with one feedback loop, then cancelling these dynamics with another. In fact it is odd, but it must be understood that the emphasis here is not on manipulator control, but rather on the use of the manipulator plus control to represent a generalized second-order system. In particular, this has been done to demonstrate that two control systems with identical loop transfer functions may have significantly different driving point impedances. Toward this end, three different PID controllers have been implemented to compensate for three different “plants”.

For each link, the control is of the form:

\[
U = U_1 + U_2 \\
U_1 = -K\theta - B\omega \\
U_2 = \frac{1}{\tau}(B + \frac{K}{s} + \frac{Mas}{s + a})e
\]

where \(e = \theta_0 - \theta\). \(U_1\) is the control which places the two poles. \(U_2\) is the PID control; \(a\) sets the bandwidth of the derivative compensation. Of course, these controllers are digitally implemented, and therefore must be converted to the \(z\)-domain:

\[
U_{1k} = -K\theta_k - B\omega_k \\
U_{2k} = \frac{1}{\tau}(B + \frac{KT}{z-1} + Ma\frac{z-1}{z-\delta})e_k
\]
where \( k \) is the discrete time index, \( T \) is the sampling period, and \( \delta = e^{-aT} \). For ease of implementation, these equations are converted to state space form:

\[
\begin{bmatrix}
\mu_{k+1} \\
\nu_{k+1}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\delta & 1 + \delta
\end{bmatrix}
\begin{bmatrix}
\mu_k \\
\nu_k
\end{bmatrix} +
\begin{bmatrix}
0 & 0 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
\omega_k \\
\omega_k
\end{bmatrix}
\]

\[
U_k =
\begin{bmatrix}
\beta_1 & \beta_2
\end{bmatrix}
\begin{bmatrix}
\mu_k \\
\nu_k
\end{bmatrix} +
\begin{bmatrix}
-(\beta_3 + K) & -B
\end{bmatrix}
\begin{bmatrix}
\omega_k \\
\omega_k
\end{bmatrix}
\]

where:

\[
\beta_1 = (Ma(1 - \delta) - KT\delta)/\tau
\]
\[
\beta_2 = (KT - Ma(1 - \delta))/\tau
\]
\[
\beta_3 = (B + Ma)/\tau
\]

and where \( \mu \) and \( \nu \) are the states of the compensator. If these equations are treated as those of link 1, and similar equations for link 2 are augmented, the result may be written in the following compact notation:

\[
\chi_{k+1} = \Phi_c \chi_k + \Gamma_c \Theta_k
\]
\[
U_k = C_c \chi_k + D_c \Theta_k
\]

where:

\[
\chi' =
\begin{bmatrix}
\mu_1 & \mu_2 & \nu_1 & \nu_2
\end{bmatrix}
\]
\[
\Phi_c =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\delta_1 & 0 & 1 + \delta_1 & 0 \\
0 & -\delta_2 & 0 & 1 + \delta_2
\end{bmatrix},
\Gamma_c =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}
\]
\[
C_c =
\begin{bmatrix}
\beta_{11} & 0 & \beta_{21} & 0 \\
0 & \beta_{12} & 0 & \beta_{22}
\end{bmatrix},
D_c =
\begin{bmatrix}
-(\beta_3 + K_1) & 0 & -B_1 & 0 \\
0 & -(\beta_3 + K_2) & 0 & -B_2
\end{bmatrix}
\]

This is the form in which the compensator is implemented.

At this point it would be convenient to examine the closed loop Nyquist plot of the driving point admittance. There is a problem, however, in that the compensator is
discrete time, and the plant is continuous time. The plant model may be discretized, but it must be understood that the driving point impedance is fundamentally a continuous time function, regardless of the control implementation.

This suggests that a continuous time approach is appropriate. However, while simulation is possible, an exact analytical solution for the driving point impedance of a continuous time plant with a discrete time controller does not exist. Of course, a continuous time approximation to the discrete time controller can be used, but it would be useful to investigate the effects of the finite sampling period.

One approach is to discretize the plant assuming a zero-order hold, which is correct at the control input, but incorrect at the environmental input. This approach leads to the following closed loop state equations:

\[
\begin{bmatrix}
\Theta_{k+1} \\
\chi_{k+1}
\end{bmatrix} = \begin{bmatrix}
\Phi + \Gamma_B D_c & \Gamma_B C_c \\
\Gamma_c & \Phi_c
\end{bmatrix} \begin{bmatrix}
\Theta_k \\
\chi_k
\end{bmatrix} + \begin{bmatrix}
\Gamma_L \\
0
\end{bmatrix} F_k
\]

\[V_k = \begin{bmatrix}
0 & J \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\Theta_k \\
\chi_k
\end{bmatrix}\]

where \( V \) is the cartesian endpoint velocity of the manipulator, and:

\[
\Phi = e^{AT}
\]

\[
\Gamma_B = \int_0^T e^{A\sigma} d\sigma B
\]

\[
\Gamma_L = \int_0^T e^{A\sigma} d\sigma L
\]

where \( A, B, \) and \( L \) are the matrices defined in equation 4.2. Defining \( \Sigma' = [\Theta' \chi'] \), the state equations can be written in a compact form:

\[
\Sigma_{k+1} = \Phi_d \Sigma_k + \Gamma_d F_k
\]

\[V_k = C_d \Sigma_k\]

The discrete time admittance is now defined as:

\[Y^*(z) = C_d (zI - \Phi_d)^{-1} \Gamma_d\]
The frequency response plots of discrete time systems generated by commercial software packages involve computation of the gain and phase of $Y^*(e^{j\omega T})$ for $\omega$ in the range $0 < \omega < \omega_N$, where $\omega_N$ is the Nyquist frequency. If a Bode or Nyquist plot is generated in such a fashion, the question becomes, how does the result relate to the continuous time admittance?

The procedure is to assume that $Y(s)$ exists, as indicated in Figure 4.12, imbedded between a zero-order hold and a sampler, and that the collection is equivalent to $Y^*(z)$. Thus, to extract the frequency response of $Y(s)$ from that of $Y^*(z)$, we need only subtract the effects of the zero-order hold and the sampler. The hold has the following transfer function:

$$H_{ZOH}(j\omega) = T \left[ \sin \left( \frac{\omega T}{2} \right) \right] e^{-j\left(\frac{\omega T}{2}\right)}$$

The sampler has a transfer function of $1/T$. It is straightforward to find the magnitude and the phase of the product of these transfer functions at any frequency $\omega$, and to subtract these from the overall frequency response to arrive at that of $Y(s)$.

A drawback to this method, however, is that $Y^*(z)$ is meaningful only at frequencies less than the Nyquist frequency ($\pi/T$ rad/sec), so that the high frequency behavior of the impedance is unavailable. For the experiments described in the next chapter, all interesting phenomena occur at low enough frequencies that this is not an issue. This,
therefore, is the analytical approach that has been taken.

At frequencies well below the Nyquist frequency, the combination of zero-order hold and sampler behaves simply as a delay of $T/2$, so that the magnitude scaling is unity, and the phase lag is $\omega T/2$ radians. For the examples presented in this thesis, the Nyquist frequency is 50 Hz and the maximum frequency of interest is 8 Hz, so that it is a simple matter to correct the product of the digital analysis by subtracting the phase lag $\omega T/2$.

This analysis was performed for three different controllers, each designed to compensate for a different second-order plant. The relevant parameters for the three designs are:

Case 1 (subscripts refer to link number):

\[
\begin{align*}
\tau_1 &= \tau_2 = 0.2 \text{ sec} \\
a_1 &= a_2 = 75.0 \text{ rad/sec} \\
M_1 &= 0.1869 \text{ kg-m}^2 \\
M_2 &= 0.0860 \text{ kg-m}^2 \\
K_1 &= 25.0 \text{ kg-m}^2/\text{s}^2 \\
K_2 &= 11.5 \text{ kg-m}^2/\text{s}^2 \\
B_1 &= 2.00 \text{ kg-m}^2/\text{s} \\
B_2 &= 0.92 \text{ kg-m}^2/\text{s}
\end{align*}
\]

Case 2 (same as Case 1 except as noted):

\[
\begin{align*}
B_1 &= 1.00 \text{ kg-m}^2/\text{s} \\
B_2 &= 0.46 \text{ kg-m}^2/\text{s}
\end{align*}
\]

Case 3 (same as Case 1 except as noted):

\[
\begin{align*}
B_1 &= 0.50 \text{ kg-m}^2/\text{s} \\
B_2 &= 0.23 \text{ kg-m}^2/\text{s}
\end{align*}
\]
Figure 4.13: Driving point admittance, $Y_{zz}(s)$. All three PID controlled systems. Both axes have units of sec/kg.

The expectation is that these three designs will exhibit similar servo behavior, but due to the different selections of $B_1$ and $B_2$, different coupled stability properties. This is supported by the Nyquist plots of $Y_{zz}(s)$ and $Y_{yy}(s)$ shown in Figures 4.13 and 4.14. Although all three designs violate the coupled stability criterion, the masses which will lead to instability will be substantially different in the different cases.
Figure 4.14: Driving point admittance, $Y_{yy}(s)$. All three PID controlled systems. Both axes have units of sec/kg.
4.4 LQG/LTR Control

This section describes an LQG/LTR controller which is implemented on the two-link manipulator. The LQG/LTR method is essentially a multivariable servo design technique; it is described in greater detail in Appendix B. For the purposes of this section, it is important only to understand the following points:

- This implementation of LQG/LTR includes integral action in each control channel.

- The primary goal of the procedure is to "shape" the frequency domain behavior of the singular values of the loop transfer function. In particular, this implementation attempts to "match" the magnitudes of all the singular values, and to make these have a constant slope of \(-20 \text{ dB/decade}\) in the frequency domain, at least for some desired bandwidth. Furthermore, this is done by generating a plant inversion. In other words, this is just a multivariable generalization of the PID strategy presented in the last section.

- To emphasize the multivariable nature of the design, the nominal manipulator configuration is chosen not to be decoupled. The link angles are: \(\theta_1 = 30^\circ\), \(\theta_2 = 150^\circ\).

- Because the formal LQG/LTR design procedure described in Appendix B is ill-conditioned for a double-integrator plant (which the manipulator very nearly is), position and velocity feedback are used to move the plant poles into the LHP.

The LQG/LTR compensator for this problem is eighth order. The closed loop singular values, assuming no errors in the plant model, are shown in Figure 4.15. The LQG/LTR design procedure is continuous time, but it typically results in high order compensators which require digital implementation, so singular values corresponding
Figure 4.15: Closed loop transfer function singular values, LQG/LTR control. Top: continuous time compensator. Bottom: discrete time compensator.

to both cases are shown. Evidently, the discrete time implementation degrades performance slightly, but both plots have the classic shape of a good command-following control system.

A Nyquist plot of the driving point admittance (discrete time compensator) is shown in Figure 4.16, and Bode plots are shown in Figure 4.17. It is evident that the coupled stability property is not achieved. The large left half plane loop in the Nyquist plot occurs at low frequency and is due to integral action; it suggests stability problems upon coupling to masses. The smaller loop occurs for a range of higher frequencies (5.6
Figure 4.16: Nyquist plot of closed loop admittance \(Y_{uv}(s)\). LQG/LTR control. Both axes have units of \(\text{sec/kg}\).

Hz – 9.5 Hz), and suggests stability problems upon coupling to springs with stiffnesses within a certain range.

This controller is intended to have good servo behavior, not necessarily good interactive behavior. The point of introducing it is to show that the two goals are quite distinct. An added benefit of this implementation is that it is also an example of a controller without force feedback that can exhibit instability upon contact with a stiff surface. It is important to understand that contact instability is a consequence of the driving point impedance, not necessarily of force feedback.
Figure 4.17: Bode plots of closed loop admittance ($Y_{yy}(s)$). LQG/LTR control.