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Numerical methods for contact between two joined quarter spaces and a rigid sphere

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1. Introduction

The Hertz theory of contact is based on the assumption of a homogeneous half space. However, many engineering systems such as rail/wheel contact, roller bearing contact or contact of welded bodies pose a challenge to the extension of contact theories for more complex considerations. The contact analyses for quarter space related problems are of great importance in both theoretical research and practical applications. Hetenyi (1970) employed an iterative scheme to calculate the stress in an elastic quarter space subjected to a concentrated normal load on its surface. The key to this iterative scheme utilizes two overlapped symmetrically loaded half spaces; numerical methods are employed to free the interface plane from normal stress. A coupled pair of integral equations with an unknown pressure can be derived from the two overlapped half spaces. Keer et al. (1983) used the Fourier transform to solve the integral equations and extended the solution approach to an elastic quarter space subjected to tractions on its surface. Furthermore, Hanson and Keer (1990) used a direct method to solve the integral equations, in which arbitrary loading could be considered. Moreover, the contact behavior near the edge of a quarter space is complex, and the edge effects arising from a quarter space during contact were examined by numerical approaches (Hanson and Keer, 1991, 1995; Hanson et al., 1994) or experiments (Chai and Lawn, 2007; Gogotsi and Mudrik, 2009; Mohajerani and Spelt, 2010). On the other hand, the finite element method (FEM) was employed by Bower et al. (1987) to analyze the plastic deformation of a quarter space under rolling contact loads. The contents of the above literature were limited to problems associated with a single quarter space. The contact problems of two joined quarter spaces, i.e., welded materials, were not involved. A quarter space can be treated as a special case of two joined quarter spaces, by setting Young’s modulus of one of the materials to be zero.

The present analysis uses the equivalent inclusion method (EIM) originally proposed by Eshelby (1957) to investigate the contact between a rigid sphere and two joined quarter spaces. One of them is regarded as an inhomogeneity, which is treated by an equivalent inclusion of the same material constants as the other (cf. Eshelby (1957) and Mura (1993)) but with proper eigenstrain distribution. The entire displacement or stress fields are obtained by the superposition of the homogeneous half space solutions and the disturbed solutions due to the equivalent eigenstrains. In numerical procedure, contact pressure is approximated as piecewise constant over rectangular patches. The homogeneous half space solutions for the elastic displacements or stresses caused by the uniform pressure distributed on a rectangular area were obtained by Love (1929), Johnson (1985), Kalker (1986), Ahmadi et al. (1987), Hills et al. (1993), Liu and Wang (2002).

In the proposed modeling, analytical solutions for inclusions can be obtained following the direction of many known works, such as those for the elastic fields in a half space caused by...
spherical thermal inclusions (Mindlin and Cheng, 1950b), ellipsoidal inclusions (Seo and Mura, 1979), or cuboid inclusions (Chiu, 1978). Note that the eigenstrains given in the above references were assumed to be uniform. On the other hand, numerical approaches built upon these elementary solutions can make the analysis applicable to more general cases. The elastic fields due to eigenstrains can be expressed in terms of Galerkin vectors (Mindlin and Cheng, 1950a), where the basic Galerkin vectors in a half space were derived by Yu and Sanday (1991a,b). Based on the Galerkin vectors, Liu and Wang (2005) initiated the study of the stress fields due to eigenstrains in a half space; the full set of analytical solutions for the displacements and stresses were achieved in Liu et al. (2012). Furthermore, Liu et al. (2012) derived the influence coefficients in explicit closed-form for numerical implementation, when the computational domain is divided into a number of elementary cuboids with uniform eigenstrains. The resultant elastic field caused by all such eigenstrains was the sum of solutions contributed by each individual inclusion, where the computation highlights a seamless implementation of the three-dimensional fast Fourier transform (FFT) algorithms (Liu et al., 2000). The current work relies heavily on this method to accelerate the calculation.

The equivalent eigenstrains chosen to replace the inhomogeneities need to be determined in advance. Recently, Chen et al. (2010) and Zhou et al. (2011a,b) used the Conjugate Gradient Method (CGM) to solve the equivalent eigenstrains. Chen et al. (2010) used this method to analyze the elasto-plastic contact on a layered half space. In their model, the layered material was treated as an inhomogeneity. Furthermore, the contact problems of a single inhomogeneity or a stringer of inhomogeneities in a half space subjected to an indentation was investigated by Zhou et al. (2011a). On the other hand, spherical inhomogeneities (Leroux et al., 2010) and cylindrical inhomogeneities (Leroux and Nélias, 2011) were also investigated. However in their numerical approach, the methods used to solve the disturbed solutions due to eigenstrains were based on Zhou et al. (2009), which is an approximate method possessing difficulties in numerical error control. The computational domain needs to be extended significantly and the surface mesh should be fine enough to capture accurately the disturbance behavior of near surface inclusions. However, when a fine surface mesh is used, such an indirect method would experience extra difficulties in developing explicit relations between the redundant surface traction and the unknown eigenstrain. The present contact analysis of the two joined quarter spaces model is based on the EIM theories employing the explicit solutions of the eigenstress and the influence coefficients developed by Liu et al. (2012), which, in contrast, have circumvented the above mentioned numerical difficulties.

2. Theoretical description

2.1. Equivalent inclusion method for joined quarter spaces

The contact model for a rigid sphere indenting two joined quarter spaces is shown schematically in Fig. 1. The two quarter spaces, denoted as region 1 and region 2 with different material properties, are perfectly "welded" together. Region 2 can be treated as an inhomogeneity (denoted by \( \Omega \)) with respect to the region 1. When the two joined quarter spaces are subject to an external load, elastic fields will be disturbed by the inhomogeneity. The inhomogeneity can be replaced by an inclusion using properly chosen eigenstrains (Eshelby, 1957). This method is called equivalent inclusion method (EIM). The stress field of the two joined quarter spaces, shown in Fig. 2 (a), can be express as follows, using Hooke’s law,

\[
\sigma_q = C_{ijkl}(e_{0q} + \tilde{\varepsilon}_q) = C_{ijkl}(\epsilon_{0q} + \tilde{\varepsilon}_q) \quad \text{in } \Omega
\]

\[
\sigma_q = C_{ijkl}(e_{0q} + \tilde{\varepsilon}_q) \quad \text{in } D - \Omega
\]

where \( C_{ijkl} \) and \( C_{ijkl} \) denote the elastic moduli of the matrix \( (D - \Omega) \) and inhomogeneity \( (\Omega) \), respectively; \( \epsilon_{0q} \) is the homogeneous (i.e. in the absence of material inhomogeneity) contact solution caused by the surface pressure; and \( \tilde{\varepsilon}_q \) is the perturbed strain caused by the
inhomogeneity. Proper determination of eigenstrain $\varepsilon_{kl}$ should be employed to account for the influence of the inhomogeneity. Hooke's law can also be expressed as follows,

$$e_{0}^{kl} + \tilde{\varepsilon}_{kl} = C_{klmn}^{1} \sigma_{mn} + \varepsilon_{kl}^{0} \quad \text{in } \Omega$$  \(2\)

Substituting Eq. (2) into (1), we have,

$$\sigma_{ij} = C_{ijkl}^{1} \left( C_{klmn}^{1} \sigma_{mn} + \varepsilon_{ij}^{0} \right) \quad \text{in } \Omega$$  \(3\)

On the other hand, the stress can be written as the sum of homogeneous contact stress $\sigma_{ij}^{0}$ due to pressure and shear tractions, and eigenstress $\sigma_{ij}^{e}$ due to eigenstrain, i.e.

$$\sigma_{ij} = \sigma_{ij}^{0} + \sigma_{ij}^{e} \quad \text{(4)}$$

Substituting Eq. (4) into (3) yields

$$\sigma_{ij}^{0} + \sigma_{ij}^{e} = C_{ijkl}^{1} \left( C_{klmn}^{1} \sigma_{mn}^{0} + \sigma_{mn}^{e} \right) \quad \text{in } \Omega$$  \(5\)

or equivalently,

$$C_{ijkl}^{1} \sigma_{mn}^{0} - \sigma_{ij}^{0} + C_{ijkl}^{1} \varepsilon_{kl}^{0} = -C_{ijkl}^{1} C_{klmn}^{1} \sigma_{mn}^{0} + \sigma_{ij}^{e} \quad \text{in } \Omega$$  \(6\)

The above equation may be represented in matrix form

$$\left( C^{-1} \cdot I \right) \cdot \sigma^{e} + C \cdot \varepsilon^{e} = \left( I - C^{-1} \right) \cdot \sigma^{0}$$  \(7\)

where $I$ is the unit matrix.

The eigenstress $\sigma_{ij}^{e}$ due to eigenstrain in a half space can be expressed as $\left(\text{Liu et al. (2012)}\right)$.

### Table 1. Material parameters and contact conditions.

<table>
<thead>
<tr>
<th>Parameter value</th>
<th>Normal load, $W$ (N)</th>
<th>Friction coefficient, $\mu_{f}$</th>
<th>Young's modulus in left quarter space, $E_{1}$ (GPa)</th>
<th>Poisson's ratio in left quarter space, $\nu_{1}$</th>
<th>Maximum Hertzian contact pressure, $p_{h}$ (MPa)</th>
<th>Hertzian contact radius, $a$ (mm)</th>
<th>Distance, $d/a$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1923</td>
<td>0.0, 0.2, 0.4, 0.6</td>
<td>210</td>
<td>0.3</td>
<td>3672.76</td>
<td>0.5</td>
<td>0 ~ 3</td>
</tr>
</tbody>
</table>


Fig. 1. Contact model and coordinate system illustrated by a rigid sphere sliding on two joined quarter spaces, assuming perfect bonding along the common boundaries without any crack at the joint interface.

Fig. 2. Equivalent inclusion method for joined half spaces: (a) Stress field of two joined quarter spaces ($C_{ijkl}^{1}$ and $C_{ijkl}^{2}$ denote the elastic moduli for the matrix ($D - \Omega$) and inhomogeneity ($\Omega$)), (b) Elemental cuboids and element size.
\[
\sigma_0^y(x) = \frac{-\mu}{4\pi(1-v)} \int_{\Omega} \left( \Theta_y - \overline{R}_y - 2z\overline{R}_{3y} + 2z^2\delta_y \right) \mathbf{e}^y \, dx
\]

in \( D - \Omega \) (8a)

\[
\sigma_0^x(x) = \frac{-\mu}{4\pi(1-v)} \int_{\Omega} \left( \Theta_x - \overline{R}_x - 2z\overline{R}_{3x} + 2z^2\delta_x \right) \mathbf{e}^x \, dx
\]

\(-2\mu v^y - \frac{2\mu}{1-v} v^e \delta_y \) in \( \Omega \) (8b)

Fig. 3. Modeling flowchart.

Fig. 4. Verification of the present method with the FEM. (a) FEM model with mesh design, (b) Pressures along the \( x \)-axis for \( E_2 = 4E_1 \) and \( E_2 = 0 \), (c) Stresses for \( E_2 = 4E_1 \) along the line through point (0,0,0.72a) and parallel to the \( x \)-axis, (d) Stresses for \( E_2 = 0 \) along the line through point (0,0,0.72a) and parallel to the \( x \)-axis.
where $\mathbf{e'} = [e_{11}; e_{22}; e_{33}; 2e_{12}; 2e_{13}; 2e_{23}]^T$, vectors $\mathbf{R}$, $\mathbf{R}$ and $\phi$ are the second order derivatives of potentials (Liu et al. (2012)), and the comma indicates partial differentiation.

In the numerical simulation, the two quarter half spaces, shown in Fig. 2 (b), are meshed into several constant-sized cuboids. The sides of the cuboid have lengths of $2\Delta x$, $2\Delta y$ and $2\Delta z$ in their respective $x$, $y$, $z$ directions, and each cuboid is assumed to have uniform eigenstrains. After discretization, Eq. (8) can be re-written as cuboidal contributions.
expressed in a simplified form the influence coefficients and symbol ‘’ sent convolution and correlation-convolution, Eq. (9) can be able in Liu et al. (2012), Appendix A2. Using the notation and FFT (DC-FFT) for the first term, combine 3D discrete convolution-correlation combination terms. When solving the multiple and eigenstrain to eigenstress. Eq.(9) includes convolution and convolution (due to the need of boundary condition satisfaction): (a) Pressure along x-axis, (b) Von Mises stress along the z-axis, (c) CPU time consumption by the present method and FEM, (d) Memory usage by the present method and FEM.

\[
\sigma_{ij}^{r}(\alpha, \beta, \gamma) = \frac{-\mu}{4\pi(1-\nu)} \left( \sum_{\alpha=0}^{L-1} \sum_{\beta=0}^{N-1} \sum_{\gamma=0}^{M-1} T_{ijkl}^{(0)}(\alpha - \xi, \beta - \eta, \gamma - \vartheta) e_{ijkl}(\xi, \eta, \vartheta) \right)
\]

\[
+ \sum_{\alpha=0}^{L-1} \sum_{\beta=0}^{N-1} \sum_{\gamma=0}^{M-1} T_{ijkl}^{(1)}(\alpha - \xi, \beta - \eta, \gamma + \vartheta) e_{ijkl}(\xi, \eta, \vartheta)
\]

\[
+ \sum_{\alpha=0}^{L-1} \sum_{\beta=0}^{N-1} \sum_{\gamma=0}^{M-1} T_{ijkl}^{(2)}(\alpha - \xi, \beta + \eta, \gamma + \vartheta) e_{ijkl}(\xi, \eta, \vartheta)
\]

\[
+ z^2 \sum_{\alpha=0}^{L-1} \sum_{\beta=0}^{N-1} \sum_{\gamma=0}^{M-1} T_{ijkl}^{(3)}(\alpha - \xi, \beta - \eta, \gamma + \vartheta) e_{ijkl}(\xi, \eta, \vartheta)
\]

\[
(0 \leq \alpha \leq M - 1, 0 \leq \beta \leq N - 1, 0 \leq \gamma \leq L - 1)
\]

(9)

where \(T_{ijkl}^{(0)}\), \(T_{ijkl}^{(1)}\), \(T_{ijkl}^{(2)}\) and \(T_{ijkl}^{(3)}\) are the influence coefficients relating eigenstrain to eigenstress. Eq. (9) includes convolution and convolution-correlation combination terms. When solving the multiple cuboidal eigenstrain problem, one can conduct 3D discrete convolution and FFT (DC-FFT) for the first term, combine 3D discrete convolution-discrete correlation and FFT (DC-DCR-FFT) (DC-FFT in the x and y directions and DCR-FFT in z direction) for the others. More details on the influence coefficients \(T_{ijkl}^{(0)}\), \(T_{ijkl}^{(1)}\), \(T_{ijkl}^{(2)}\) and \(T_{ijkl}^{(3)}\) are available in Liu et al. (2012), Appendix A2. Using the notation \(A\) for the influence coefficients and symbol “” for an operator to represent convolution and correlation-convolution, Eq. (9) can be expressed in a simplified form

\[
\sigma^{r} = A \otimes \varepsilon^{r}
\]

(10)

Submitting (10) into (7) leads to

\[
(C^{-1} - I) \cdot A \otimes \varepsilon^{r} + \varepsilon^{r} = (I - C C^{-1}) \cdot \sigma^{0}
\]

(11)

Eq. (11) is a set of linear equations with unknown eigenstrain \(\varepsilon^{r}\). If the grids of \(M_{1} \times N_{1} \times L_{1}\) are employed to mesh the inhomogeneity (see Fig. 2(b)), the above system of \(6 \times M_{1} \times N_{1} \times L_{1}\) equations for \(6 \times M_{1} \times N_{1} \times L_{1}\) unknown \(\varepsilon^{r}\) may be solved by using CGM (Zhou et al. (2011a,b)).

When the eigenstrains are obtained, the surface perturbed displacement \(\tilde{u}_{i}\) is given as follows (Liu et al. (2012)):

\[
\tilde{u}_{i} = -\frac{1}{2\pi} \int_{\partial} U_{i}^{s}(\varepsilon) d\mathbf{x}
\]

(12)

The explicit expression of \(U_{i}^{s}\) can be found in Liu et al. (2012). Similarly, after discretization, Eq. (12) can be solved by using 2D DC-FFT.

2.2. Contact model

The boundary conditions for two bodies in contact can be described by the following system of equations and inequalities as in most numerical contact models:

\[
p(x, y) \geq 0, \quad g(x, y) = 0 \quad \text{in contact regions}
\]

(13a)

\[
p(x, y) = 0, \quad g(x, y) \geq 0 \quad \text{in contact regions}
\]

(13b)

\[
\iint p(x, y) dxdy = W
\]

(13c)

where \(p\) denotes pressure and \(W\) is the normal load; \(g\) the surface gap consisting of \(h_{0}\) the initial body separation between the two surfaces, \(\delta_{z}\) the z direction rigid displacement, and \(u_{z}\) the surface normal displacement. The contact bodies are meshed into a number

Fig. 6. Grid test and comparisons of the CPU time and computer memory usage by the present method and FEM for different mesh densities (The simulation domain is \([-1.5a, 4.5a] \times (-3a, 3a) \times (0, 4a)\) along the x, y and z directions for the present method and \((-10a, 10a) \times (-10a, 10a) \times (0, 10a)\) for FEM. The larger domain for the latter is due to the need of boundary condition satisfaction): (a) Pressure along x-axis, (b) Von Mises stress along the z-axis, (c) CPU time consumption by the present method and FEM, (d) Memory usage by the present method and FEM.
of equal-sized cuboids; thus the surface gap can be written in the following matrix form,
\[ g = \frac{h_0}{C_0} dz + u_z \quad (14) \]

When the contact materials contain inhomogeneities, the equivalent inclusion method is used to treat inhomogeneities as inclusions with appropriate eigenstrains. The surface displacement can be expressed as the sum of \( u_e_z \) the surface elastic displacement (a homogeneous half space solution) and \( \tilde{u}_z \) the perturbed normal displacement on surface due to inhomogeneity. That is
\[ u_z = u_e_z + \tilde{u}_z \quad (15) \]

Since \( u_e_z \) is the half space solution, it can be calculated in terms of influence coefficients by using the FFT method indicated below.
\[ u_e_z = \text{IFFT} \left( \mathbf{C}_{y_0} \ast \mathbf{q}_y \right) + \text{IFFT} \left( \mathbf{C}_{y_0} \ast \mathbf{q}_y \right) + \text{IFFT} \left( \mathbf{C}_p \ast \mathbf{p} \right) \quad (16) \]

where the asterisk is the symbol for convolution; \( \mathbf{q}_y \) and \( \mathbf{q}_z \) are the shear tractions along the \( x, y \) directions, respectively; \( \mathbf{C}_{y_0}, \mathbf{C}_y, \mathbf{C}_p \) are related influence coefficients for the half space solutions depending on the elastic properties of the material of region 1 and the mesh size. In an elastic contact of homogeneous material, \( \mathbf{u}_z = \mathbf{0} \).

Eq. (14) is a set of linear equations; the CGM is used to accelerate the solution for pressure with constraints on Eq. (13). For more details about the iteration scheme, the reader may refer to Polonsky and Keer (1999). The flowchart for solving the contact problem of inhomogeneous materials is shown in Fig. 3.

3. Numerical results and discussion

In this section, the contact between a rigid sphere and a composite half space of two joined quarter spaces (Fig. 1), is analyzed. Normal load \( W \) and tangential load \( F_x \) are applied to the rigid sphere. Frictional contact is considered by assuming a surface friction coefficient \( \mu \). The contact center is located at the origin of coordinates, and parameter \( d \) is the distance between the contact center and the joined interface between the two quarter spaces. The Young’s modulus value \( E_1 \) for region 1 is kept constant, while the Young’s modulus value \( E_2 \) of region 2 is varied. The material parameters and contact conditions are listed in Table 1. In the following simulation, the pressure (or stresses) and displacements are normalized, respectively, by the maximum Hertzian pressure \( p_h \) and contact radius \( a \) from the corresponding homogeneous contact case of the material of region 1. The simulation domain is \((-1.5a, 4.5a) \times (-3a, 3a) \times (0, 4a)\) along the \( x, y \) and \( z \) directions and is discretized uniformly into \( 64 \times 64 \times 50 \) cuboids. The material inhomogeneity outside the simulation domain is neglected and is therefore treated as homogeneous with the same elastic properties as region 1. This assumption will be validated in the following discussion.

3.1. Model verification

The results obtained from current method are compared to those from FEM. A personal computer with 2.5 GHz and i5 CPU was used to calculate these cases. The commercial software
Fig. 9. Dimensionless von Mises stress for different $d$: (a) $E_2 = 4E_1$ and along the contact axis, (b) $E_2 = 0$ and along the contact axis, (c) Relationship between the maximum von Mises stress and $d$ for $E_2 = 4E_1$, (d) Relationship between the maximum von Mises stress and $d$ for $E_2 = 0$.

Fig. 10. Dimensionless load-displacement curves for different $d$: (a) $E_2 = 4E_1$, (b) $E_2 = 0$.

Fig. 11. Dimensionless pressure $p/p_h$ along the $x$-axis for different $E_2$ and $\mu_f$ when $d = 1.2a$: (a) $E_2 = 4E_1$, (b) $E_2 = 0$. 
package ABAQUS v6.5 was used to conduct the three-dimensional finite-element analysis for the same problem. The mesh for the two joined quarter spaces is plotted in Fig. 4(a) and elements in the mesh are type C3D8R (Eight-node linear brick with reduced integration are hourglass controlled). The simulations were carried out with a fixed dimensionless distance \( d/a = 1.2 \) and friction coefficient \( \mu_l = 0 \). The contact pressures along the \( x \)-axis are plotted in Fig. 4(b) for \( E_2 = 4E_1 \) and \( E_2 = 0 \). Fig. 4(b) reveals that for the case of \( E_2 = 4E_1 \), the pressure profile shifts toward the harder quarter space, i.e., the positive direction of the \( x \)-axis, while for \( E_2 = 0 \), the contact region moves toward the negative direction of the \( x \)-axis.

Fig. 4(c) and (d) give the stresses for \( E_2 = 4E_1 \) and \( E_2 = 0 \), respectively. The von Mises stresses show discontinuities at the joined interface between the two quarter spaces. Furthermore, contour plots of the von Mises stresses in the \( y = 0 \) plane are presented in Fig. 5 for \( E_2 = 4E_1 \), \( E_2 = E_1 \), \( E_2 = 0.5E_1 \) and \( E_2 = 0 \). The results obtained by the current method are in good agreement with those obtained by FEM. The results also indicate that the simulation domain is sufficient to reduce the errors induced by the homogeneity assumption outside of the simulation domain.

Furthermore, Fig. 6(a) and (b) show the mesh density test for pressure and stress, respectively. Although the results can be improved by increasing mesh density, the mesh of \( 64 \times 64 \times 50 \) is sufficiently accurate for the present numerical studies. Fig. 6(c) and (d) are the comparisons of the CPU time and computer memory between the present method and FEM for different mesh densities. The present method is faster and considerably less memory consuming than FEM. Note that a moderate simulation domain in FEM analysis is \((-10a, 10a) \times (-10a, 10a) \times (0, 10a)\) in order to satisfy the half space boundary conditions. Thus, even under the same grid size, the present method provides more grids in the contact region than does FEM.

### 3.2. Parameter study

#### 3.2.1. Effect of \( d \)

When studying the effect of the contact location, friction coefficient \( \mu_l \) was kept to be zero, while parameter \( d \) varied from 1 to 3. The dimensionless pressures along the \( x \)-axis for different \( d \) are shown in Fig. 7. As \( d \) increases, the pressure profile moves to the left (in the direction of the more compliant quarter space) in the case of \( E_2 = 4E_1 \), while for \( E_2 = 0 \) the trend is reversed. Note that a higher pressure can be observed at the joined interface between the two quarter spaces when \( d/a = 1 \) and for \( E_2 = 4E_1 \) (Fig. 7(a)), which means both quarter spaces are directly involved in the contact; more details will be discussed in section 3.3. Fig. 8 shows the surface compressive or tensile stresses along the \( x \)-axis for different \( d \). The right side of the contact exhibits a high tensile stress in the case of \( E_2 = 4E_1 \) (Fig. 8(a)). Moreover, as \( d \) increases, the maximum value of tensional stress drops, and the curve of \( \sigma_{xx} \) becomes symmetric with respect to the y axis. For the case of \( E_2 = 0 \) (Fig. 8(b)), the maximum tensile stress lies to the left side of the contact, and increasing \( d \) makes the maximum tensile stress decrease and the maximum compressive stress increase. Again, when \( d/a \) is sufficiently large, i.e., \( d/a > 3.0 \), the effect of stresses on the right quarter space disappears.

Fig. 9(a) and (b) show the distribution of dimensionless von Mises stress along the contact axis for different values of \( d \). A small \( E_2 \) promotes a higher von Mises stress than does a large \( E_2 \). As \( d \)
increases, the von Mises stress profile moves upward in the case of $E_2 = 4E_1$ (Fig. 9(a)), while for case of $E_2 = 0$, the von Mises stress profile moves downward (Fig. 9(b)). Fig. 9(c) and (d) give the maximum von Mises stresses as a function of $d$ for $E_2 = 4E_1$ and $E_2 = 0$, respectively. Note that in the case of $E_2 = 4E_1$, when $d/a = 1.0$, the maximum von Mises stress is located in the right quarter space. Thus in this case, parameter $d/a$ is chosen from 1.1 to 3 inclusively, which makes the left quarter space the focus of this study. The maximum von Mises stress becomes larger with increasing $d$ in the case of $E_2 = 4E_1$. For example, the dimensionless maximum von Mises stress is 0.607 when $d/a = 1.1$, while it increases to 0.618 when $d/a = 3$. However for the case of $E_2 = 0$, the curve of the maximum von Mises stress declines as $d$ increases but changes more than when $E_2 = 4E_1$.

The load-displacement curves are plotted in Fig. 10 for different values of $d$. When $d$ increases, the slope of the curve decreases in the case of $E_2 = 4E_1$ (Fig. 10(a)) for a given indentation depth; while for $E_2 = 0$, the tendency is reversed (Fig. 10(b)). When $d/a = 3$, the load-displacement curve is nearly the same as that of the homogeneous solution obtained by the Hertz theory (Johnson, 1985).

### 3.2.2. Effect of $\mu_f$

In this section, sliding contact is considered, and the parameter $d$ is fixed at 1.2a, while the friction coefficient varies from 0 to 0.6. As in the partial slip case investigated by Wang et al. (2010), tangential traction can result in deformation in the normal direction, which couples the pressure and shear traction. However, in the present study, sliding friction follows Amonton’s law without considering this coupling effect. A similar simplification consequence, Goodman’s approximation, was examined in detail in Hills et al. (1993). Fig. 11(a) and (b) show the dimensionless pressure along the x-axis for different values of $\mu_f$ for $E_2 = 4E_1$ and $E_2 = 0$, respectively. The pressure curve shifts towards the trailing edge of the contact area with increasing friction coefficient similar to the result for homogeneous contact. Fig. 12 shows in-surface normal stress $\sigma_{xx}$ along the x-axis for different $\mu_f$. High friction coefficients increase the magnitude of stress $\sigma_{xx}$. Furthermore, as the friction coefficient $\mu_f$ increases, the tensile stress increases at the trailing edge of the contact area and decreases at the leading edge. The interfacial shear stresses at $x = 1.2d$ for different $E_2$ and $\mu_f$ are given in Fig. 13. The maximum interfacial shear stress increases with $E_2$, which means a high $E_2$ is prone to cause interfacial failure. Again, friction enhances the maximum interfacial shear stresses.

Fig. 14 shows the dimensionless von Mises stress along the z axis or through point (0,0,0.72a) and parallel to the x-axis with varying $\mu_f$ for $E_2 = 4E_1$ or $E_2 = 0$. The von Mises stress along the z-axis first increases, then decreases with low $\mu_f$; however it decreases monotonically when $\mu_f$ is sufficiently greater for both the $E_2 = 4E_1$ and $E_2 = 0$ cases (Fig. 14(a) and (c)). Because the two quarter spaces are assumed to be perfectly bonded together along a common edge without any crack at the joint interfaces, stress components $\sigma_{yy}$, $\sigma_{yz}$ and $\sigma_{zx}$ are discontinuous at the interface. Thus, the von Mises stress exhibits significant discontinuities across interfaces (Fig. 14(b) and (d)). Moreover, if $E_2 > E_1$, the von Mises stress at the right side of interface is larger than that at the left side.

Contour plots of dimensionless von Mises stresses in the $y = 0$ plane for various $\mu_f$ are shown in Fig. 15. Friction increases the

![Fig. 14](image-url) Dimensionless von Mises stress for different $E_2$ and $\mu_f$ when $d = 1.2a$: (a) $E_2 = 4E_1$ and along the contact axis, (b) $E_2 = 4E_1$ and through point (0,0,0.72a) and parallel to the x-axis, (c) $E_2 = 0$ and along the contact axis, (d) $E_2 = 0$ and through point (0,0,0.72a) and parallel to the x-axis.
maximum normalized von Mises stress and shifts its position towards the interfaces. For the case of $E_2 = 4E_1$, the maximum normalized von Mises stress for $\mu_f = 0.2$ is 0.631, and its position moves near to the leading edge of the contact area (see Fig. 15(a)), while for $\mu_f = 0.6$, the von Mises stress has a maximum value at the surface, but its position moves near to the trailing edge of the contact area (see Fig. 15(b)). Under the same load with same friction coefficient $\mu_f$ and parameter $d$, the maximum von Mises stress in the case of $E_2 = 4E_1$ is smaller than that in the $E_2 = 0$ case. However, a larger $E_2$ can produce a higher interfacial shear stress. Generally, the von Mises stress becomes larger with increasing friction coefficient and also when the Young's modulus of right quarter space decreases relative to the left quarter space. Furthermore, Table 2 lists the maximum normalized von Mises stress for different $d$ and various $\mu_f$.

### 3.3. Further discussion on interfacial or edge effects

When the contact occurs at both quarter spaces, the pressure becomes discontinuous at the interfaces due to the material property mismatch. In this section, frictionless contact on both quarter spaces is investigated. Parameter $d$ is selected to be zero, which means the contact center is at the joint interface. The material parameters and normal load are the same as those listed in Table 1. In order to produce more accurate solutions near the interface, the simulation domain is divided into $128 \times 128 \times 50$ cuboids along the $x$, $y$ and $z$-directions, respectively.

Fig. 16 shows the contour plot of the contact pressure distribution for the case of $E_2 = 4E_1$ and $E_2 = 0.5E_1$. For a stiffer right quarter space, the contact region is relatively smaller and the pressure is

<table>
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<tr>
<th>$E_2 / E_1$</th>
<th>$d/a$</th>
<th>$\mu_f = 0.0$</th>
<th>$\mu_f = 0.2$</th>
<th>$\mu_f = 0.4$</th>
<th>$\mu_f = 0.6$</th>
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**Fig. 15.** Contour plots of dimensionless von Mises stresses in plane $y = 0$ for different $E_2$ and $\mu_f$ when $d = 1.2a$: (a) $E_2 = 4E_1$ and $\mu_f = 0.2$, (b) $E_2 = 4E_1$ and $\mu_f = 0.6$, (c) $E_2 = 0$ and $\mu_f = 0.2$, (d) $E_2 = 0$ and $\mu_f = 0.6$. **Table 2**

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higher, while the trend reverses for the case of a more compliant right quarter space. Furthermore, the dimensionless pressure $p/p_h$ and surface stress $r_{xx}/p_h$ along the $x$-axis and the von Mises stress in plane $y = 0$ are shown in Fig. 17. Discontinuous pressure and von Mises stresses can be seen at the interfaces. The maximum pressure and surface compressive stress are found on the stiffer quarter space. Moreover, curves of the surface tensile stresses show a higher shoulder at the contact edge on the stiffer quarter space. However, the maximum von Mises stress is found at the interface on the stiffer material side. Again, a high von Mises stress at the interface is prone to cause interfacial failure.

Because the pressure distribution is not symmetric with respect to the $y$-axis, a moment is needed to hold the sphere. As for the case of $E_2 = 4E_1$, the moment $M_y$ is $331.89\, \text{N mm}$, while for $E_2 = 0.5E_1$, moment $M_y$ is $-267.07\, \text{N mm}$. The present numerical analysis indicates that the moment is negligible, provided that the separation distance $d$ is larger than $a$.

4. Conclusions

A new numerical contact method for solving joined quarter space contact problems has been developed. With this method, one of the joined quarter spaces can be treated as an inhomogeneity, and the EIM is employed to determine the equivalent eigenstrains. The displacement and stress fields are the superposition of the half space solution and the disturbed solution due to eigenstrains. The results calculated by the proposed method are in good agreement with corresponding FEM solutions.

Problems of a rigid sphere in contact with joined quarters with different Young’s moduli and subjected to various friction coefficients are analyzed. Results show that the pressure and stresses are influenced by parameter $d$, the distance between the contact center and the quarter space joint interface, and friction coefficient $\mu_f$. As the parameter $d$ increases, the pressure profile moves to the left (the direction of the more compliant quarter space) and the maximum von Mises stress increases in the case that the contact center is at the more compliant material side (i.e. $E_2 = 4E_1$), while for $E_2 = 0$, the case of a quarter space, the trend is reversed. The value of von Mises stress becomes higher with increasing friction coefficient, and when the Young’s modulus of the right quarter space decreases relative to the left quarter space, i.e., in the case of $E_2 = 0$, the maximum von Mises stress reaches the highest level. When the contact is centered at the interface of both quarter spaces, the pressure on the stiffer quarter space is higher, and the pressure and the von Mises stress are discontinuous at the joint interface.

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References


