Analytical solution for elastic fields caused by eigenstrains in a half-space and numerical implementation based on FFT

Shuangbiao Liu¹, Xiaoqing Jin, Zhanjiang Wang², Leon M. Keer, Qian Wang*

Department of Mechanical Engineering, Northwestern University, Evanston, IL 60208, USA

Abstract

Modern engineering design often faces severe challenges in accommodating impurities and imperfections of materials in the presence of considerable thermal expansion and plastic deformation. Based on micromechanics, a versatile and effective approach for such nonlinear problems can be conceived by employing an inclusion model. This paper reports on the derivation of explicit integral kernels for the elastic fields due to eigenstrains in an elastic half-space. The domain integrations of these kernels result in analytical solutions to stresses and deformations. After dividing each general kernel into four groups, the integration is resolved into three-dimensional convolutions and correlations, which can be numerically processed with algorithms based on fast Fourier transform (FFT) to enable efficient and accurate numerical computations. The analytical solution corresponding to a cuboidal inclusion (a rectangular parallelepiped domain) is obtained in an explicit closed-form and is utilized to determine influence coefficients. The present solution and numerical implementation can be used as building blocks for analyzing arbitrarily distributed thermal strains, plastic strains and material inhomogeneities, as demonstrated by solving an illustrative example of elasto-plastic contact.

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1. Introduction

Eigenstrain is the generic name, given by Mura (1993), for the nonelastic strains arising in such inelastic processes as plastic deformation, thermal expansion, phase transformation, assembly mismatch. In the context of micromechanics, the term inclusion (Eshelby, 1957) refers to a subdomain, $\Omega$, in a homogeneous material, where a non-zero eigenstrain is prescribed and the remainder of the material (matrix) is free of eigenstrain. The subdomain, $\Omega$, is called an inhomogeneity when its material properties differ from those of the matrix. It is speculated that there exists one-to-one correspondence between these two classes of problems (Li and Wang, 2008, p. 86). Mura et al. (1996) pointed out that the inclusion model provides a unified and effective treatment for a variety of classical subjects: (1) stress fields caused by nonelastic strains; (2) stress disturbances due to inhomogeneities; (3) average material properties; (4) cracks, voids, or rigid reinforcement in materials, and others. Recent micromechanical investigation of the inclusion problems has called attention to the studies involving plasticity (cf. (Guo et al., 2011; Lecarme et al., 2011; Lee et al., 2011; Kadkhodapour et al., 2011; Yassar et al., 2007)).

Because of its fundamental importance to mechanics and materials science, the inclusion problem has been frequently investigated (cf. (Mura, 1993; Mura et al., 1996)) since Eshelby’s pioneering work (Eshelby, 1957, 1959, 1961). Two basic geometries of inclusions, ellipsoidal and polyhedral, are of paramount interest for many researchers (Chiu, 1977, 1978,
1980; Jin et al., 2009a, 2010, 2011; Ju and Sun, 1999; Liu and Wang, 2005; Nozaki and Taya, 1997; Nozaki and Taya, 2001; Rodin, 1996) and are studied in both three dimensional (3D) and 2D (elliptical and polygonal). Eshelby (1957) showed that the elastic field inside an ellipsoidal inclusion under uniform eigenstrain is also uniform. However, the exterior field is much more complicated (Eshelby, 1959; Jin et al., 2011; Ju and Sun, 1999). It is also noted that both the interior and exterior stress fields of any polygonal inclusion subjected to uniform eigenstrain can be represented as a unified expression, which consists of only elementary functions (Jin et al., 2010).

The mechanics involving a surface boundary (half-space) is of significance to both engineering practice and theoretical investigation, and the corresponding inclusion problems have been studied analytically only for several specific instances. Mindlin and Cheng (1950b) solved a spherical inclusion with pure dilatational eigenstrain (thermal inclusion). Aderogba (1976) studied a spherical inclusion subjected to arbitrary eigenstrain components of uniform magnitude. Chiu (1978) used the method of images to construct a solution for a cuboidal inclusion (a rectangular parallelepiped domain), inside a half-space, with a uniform eigenstrain. Seo and Mura (1979) solved the elastic fields caused by an ellipsoidal thermal inclusion. Hu (1989) obtained explicit closed form solution for the exterior elastic field caused by a cuboidal thermal inclusion. Yu and Sanday (1990) derived axisymmetric elastic fields due to a spherical inclusion with dilatational and axial tensile eigenstrain. Wu and Du (1996) obtained an explicit solution for the displacement in a half space with a cylindrical inclusion. Liu and Wang (2005) described a general and direct method to express the stress field inside a half-space due to an arbitrary inclusion and demonstrated this method with several numerical cases. However, further progress in utilizing these solutions is needed, because explicit and closed-form expressions were not presented in their work.

The shape of the inclusion domain, \( \Omega \), and the eigenstrain distribution can be arbitrary in an engineering problem; therefore, numerical methods capable of handling a general inclusion problem have to be developed. One of the challenges in such computer-based implementations is computational efficiency. The fast Fourier transform (FFT) is a ubiquitous and powerful tool in modern computational science and technology. An abbreviated list in the book by Brigham (1988) shows seventy-seven typical application areas, including applied mechanics, numerical methods, medical imaging, signal processing, and others. Since the end of the last century, FFT techniques have emerged in a variety of tribological studies (Chen et al., 2010; Jacq et al., 2002; Ju and Farris, 1996; Liu et al., 2007; Nogi and Kato, 1997; Polonsky and Keer, 2001).

Particularly in rough-surface contact simulations (Ai and Sawamiphakdi, 1999; Ju and Farris, 1996; Nogi and Kato, 1997; Polonsky and Keer, 2000), the FFT algorithms can be seamlessly integrated with the conjugate gradient (CG) method
leading to orders of magnitude acceleration of computational speed. However, the early algorithms based on FFT encountered periodicity error (Polonsky and Keer, 2000) when solving nonperiodic mechanical problems. Some studies (Ai and Sawamiphakdi, 1999; Ju and Farris, 1996; Nogi and Kato, 1997; Polonsky and Keer, 2000) have alleviated the periodicity error with a significant domain extension, thus sacrificing computational efficiency. Liu et al. (2000) and Liu and Wang (2002) applied a discrete convolution FFT (DC-FFT) algorithm to rough contact analyses. The DC-FFT can completely eliminate the periodicity error by enforcing a wrap-around order padding on the influence coefficients together with zero-padding on the contact pressure (Jin et al., 2009c; Liu et al., 2000).

Implementation of FFT techniques in solving inclusion problems requires the elementary solution for a cuboidal inclusion (Chiu, 1977, 1978, 1980; Liu and Wang, 2005), or a rectangular inclusion in the 2D case (Chiu, 1980; Jin et al., 2009a). In these solutions, a typical formulation at any field point either inside or outside the inclusion is represented by the primitive function evaluated at each vector directed from the field point to the corners of the cuboid/rectangle (Chiu, 1977, 1980; Hu, 1989; Jin et al., 2009a; Jin et al., 2009b; Liu and Wang, 2005). For any arbitrarily shaped inclusion, the elastic field may be obtained following the principle of superposition: the inclusion domain is discretized into a number of cuboidal elements, each is assumed to be subjected to uniform eigenstrains, and the resultant contributions to any field point can be superposed. It is seen that the summation for an infinite space is a pure 3D discrete convolution, since the primitive functions (Chiu, 1977) only depend on the relative locations of the field point and the cuboidal element (also called a convolution kernel). Accordingly, the 3D DC-FFT is directly applicable for full-space inclusion problems.

For half-space related inclusion problems, Jacq et al. (2002) utilized Chiu’s half-space cuboidal inclusion solution (Chiu, 1978) to develop an elasto-plastic contact model. It is noted that the final formulation of Chiu’s solution (1978) is given in a complicated form, which resorts to recursive relations involving Legendre polynomials. Moreover, since Chiu’s formulation (1978) does not explicitly demonstrate the convolution/correlation properties in the depth direction, a corresponding FFT algorithm (Jacq et al., 2002) can only be performed as a 2D layer-by-layer computation. Recently, Zhou et al. (2009) proposed a numerically efficient approach based on the indirect (mirror-image) formulae and immediately calculated surface correction pressure, which was introduced to obtain a traction-free surface. The contribution of the surface correction pressure to the stress field is evaluated by a 2D FFT. This unknown surface pressure distribution exists at the entire surface and thus its effective truncation is not a priori known, and consequently its discretization. Therefore, numerical errors related to this issue may be difficult to control, particularly when the inclusion approaches the surface of the half-space. Liu and Wang (2005) successfully developed a hybrid numerical approach utilizing the DC-FFT and discrete correlation FFT (DCR-FFT) methods based on explicit formulae and gave example cases where eigenstrains excluded shear components. Their work develops a straightforward approach to construct an accurate and rapid numerical algorithm.

Eshelby (1957) pointed out that the solution of an inclusion problem may be employed as a convenient stepping-stone in solving an inhomogeneity problem, which is known as the equivalent inclusion method (EIM): the disturbance of an applied stress due to the presence of inhomogeneity can be simulated by an eigenstress field generated by an inclusion when the eigenstrain is chosen properly. Eshelby’s inclusion solution furnishes an elegant approach for solving an ellipsoidal inhomogeneity in an infinitely extended elastic space, which is subjected to polynomial remote stress. However, for a half-space inhomogeneity problem, it is difficult, if not impossible, to implement the EIM analytically; and one has to resort to numerical computations to determine the equivalent eigenstrains, even in the simplest case of a regularly-shaped inclusion subjected to a remote uniform stress. As demonstrated by the recent work of Chen et al. (2010) and Zhou et al. (2011), half-space inclusion solutions based on Chiu’s work and indirect formulae were applied respectively in their numerical EIM studies.

The previous developed solutions for a half-space inclusion are implicit and inefficient hence the corresponding numerical computations have to be conducted through 2D FFT in a layer-by-layer manner; or are approximate and difficult to control numerical errors. The work reported in this paper intends to develop general analytical formulae for a half-space eigenstrain problem. Particularly, we successfully explore the mathematical basis for the numerical computation of a half-space inclusion by implementing 3D FFT-based algorithms. The present study has three-fold objectives: (1) to derive the explicit integral kernels for the elastic fields due to eigenstrains in an elastic half-space; (2) to obtain the explicit and closed-form analytical solution based on the derived kernels; and (3) to demonstrate that each kernel for the displacements and stresses, although not in its entirety, consists of four groups of 3D convolution/correlation terms, and therefore to implement FFT for accurate and efficient numerical results for any eigenstrain distribution. Benchmarks are given to validate both the formulae and the corresponding algorithm; and an example of elasto-plastic contact analysis is presented to illustrate the application of the present solution.

2. Theory

An isotropic and homogeneous half-space is designated by a coordinate system Ox1x2x3, which contains a volumetric irregular region (Ω) subject to eigenstrains (εij) distributed arbitrarily (Fig. 1). Material properties are given by the Lamé constants, λ and μ, and Poisson’s ratio ν. Vector x denotes an observation or response point, while X denotes a source or excitation point. In this paper a comma in the subscript denotes derivative with respect to the observation point, e.g. Fi = ∂Fi/∂xi. Note that symbols in boldface represent vectors and the Einstein summation convention is normally applied unless otherwise noted. To save space, two constants are introduced: D = 1 − 2ν and H = 1 − ν. The elastic field (ui,σij) caused by the eigenstrains is expressed in terms of Galerkin vectors, F (Yu and Sanday, 1991a):
The Galerkin vectors are written in the form of a volumetric integral as in Yu and Sanday (1991a):

\[ F(x) = \frac{\mu}{4\pi H} \int_{\Omega} \left( 2\epsilon_{ij} \delta_{ij} - \epsilon_{ij} g_{ij} \right) dx' \]

(2)

where \( g_i \) is a basic Galerkin vector for a center of dilatation in a half-space, and \( g_{ik} \) are basic Galerkin vectors either for a double force in the \( k \) direction (\( j = k \)) or a double force in the \( k \) direction with moment in the direction normal to the \( \Omega x_k \) plane (\( j \neq k \)) in a half-space. The expressions for these Galerkin vectors can be found in either Mindlin and Cheng (1950a) or Yu and Sanday (1991b), where expressions for other types of nuclei of strain are also listed. Galerkin vectors due to unit single forces \( x \) at a half-space are written as \( \text{Mindlin and Cheng, 1950a, Yu and Sanday, 1991b} \).

Galerkin vectors due to unit double forces, double forces with moment, and a center of dilatation are written as,

\[ g_i = \left[ R' + R - 2 x_i^2 \phi - 4 H D \delta_{ij}; 0; 2 x_j R_1 + 2 D (x_1 - x_j) \psi \right]^T \]

(3a)

\[ g_2 = \left[ 0; R' + R - 2 x_i^2 \phi - 4 H D \delta_{ij}; 2 x_i R_2 + 2 D (x_2 - x_i) \psi \right]^T \]

(3b)

\[ g_3 = \left[ 0; 0; R' + (3 - 4 v) R - 2 x_i x_j \phi - 4 D x_j \psi - 4 H D \psi \right]^T \]

(3c)

where semicolons are used to identify the vector components. Here, \( R' = \sqrt{(x_1 - x')^2 + (x_2 - x')^2 + (x_3 - x')^2} \) and \( R = \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2} \) are bi-harmonic potentials; \( \phi = 1/R; \psi = \ln [R + (x_3 + x_3')] \); and \( \theta = R - (x_3 + x_3') \psi \), where \( \phi, \psi \) and \( \theta \) are all harmonic potentials. Superscript \( I \) indicates that \( (x_3 - x_3') \) is involved in the Euclidian length within the material, for example, \( \phi' = 1/R' \) with \( R' \) expressed as above. Note that \( \theta_3 = -\psi \) and \( \psi_3 = \phi \). Physically, \( R' \) is the distance between \( x \) and \( x' \), while \( R \) is the distance between \( x \) and the image of \( x' \).

Galerkin vectors due to unit double forces, double forces with moment, and a center of dilatation are written as,

\[ g_j = -\partial g_i/\partial x_i \]

(4a)

\[ g_c = \left[ 0; 0; \ln \left( R' + (x_3 - x_3') \right) + (1 - 4 v) \psi + 2 x_3 \phi \right]^T \]

(4b)

Note that the derivative in Eq. (4a) is with respect to the excitation point. According to Eq. (2), the Galerkin vectors due to the eigenstrains in the region, \( \Omega \), could be written as an integral with respect to \( x' \).

\[ F(x) = \int_{\Omega} f(x, x') dx' \]

(5)

with \( f = \mu H^{-1} (2 \epsilon_{ij} \delta_{ij} - \epsilon_{ij} g_{ij}) \). Since only \( g_1 \) has a non-zero first row and only \( g_2 \) has a non-zero second row, the first and second row of \( f \) can be written as \( e_{11} \epsilon_{11} (g_{11})_1 \) and \( e_{22} \epsilon_{22} (g_{22})_2 \), respectively. Note that \( (g_{11})_1 \) is the first row of \( g_1 \) and \( (g_{22})_2 \) is the second row of \( g_2 \).

Equations (1a-c) require the derivatives of Galerkin vectors expressed in Eq. (2) or (5) in an integral form. Since the sequence of integration and differentiation may be interchanged, the derivatives of \( f \) are essential, and one can find these basic derivatives in Liu and Wang (2005). To streamline expressions in this paper, the Voigt notation (Belytschko et al., 2000) is adopted for eigenstrain components

\[ [\epsilon] = [\epsilon_{11}; \epsilon_{22}; \epsilon_{33}; 2\epsilon_{13}; 2\epsilon_{12}; 2\epsilon_{23}]^T \]

(6)
Note that one can use \( (x_3 + x_3') y = R - \theta \) to simplify \( f_{ij} \) in Eq. (10d) of Liu and Wang (2005)

\[
f_{ij} = \frac{\mu v e_i}{2 \pi H} (\phi' + (3 - 4v) \phi + 2x_3 \phi) \left( \begin{array}{c}
R'_{11} + (3 - 4v)R_{11} - 4HD\theta_{11} + 2x_3 R_{113} - 2x_3^2 \phi_{11} \\
R'_{12} + (3 - 4v)R_{12} - 4HD\theta_{12} + 2x_3 R_{123} - 2x_3^2 \phi_{12} \\
R'_{13} + (3 - 4v)R_{13} - 4HD\theta_{13} + 2x_3 R_{133} - 2x_3 \phi_{13} - 8Hx_3^2 \phi_{13} \\
R'_{23} + (3 - 4v)R_{23} + 4Hx_3^2 \phi_{23} - 2x_3 R_{233} + 2x_3^2 \phi_{23} \\
R'_{13} + (3 - 4v)R_{13} + 4Hx_3 \phi_{13} - 2x_3^2 R_{133} + 2x_3 \phi_{13} \\
R'_{12} + (3 - 4v)R_{12} - 4HD\theta_{12} + 2x_3 R_{123} - 2x_3^2 \phi_{12} \\
\end{array} \right)^T \left[ \varepsilon \right]
\]

Superscript I also indicates the initial infinite-space solution. Based on the derivatives in Liu and Wang (2005), it is straightforward to obtain higher derivatives. When the eigenstrains are plastic strains, their trace is zero, i.e., \( e_{06} = 0 \). For generality, this work deals with combinations of arbitrary eigenstrains and the terms related to the trace are therefore retained. All components of the elastic field are explicitly presented in an integration form in the following four sections.

2.2. Displacements

Three vectors characterizing the second order derivatives of the potentials are defined as follows for further simplification of expressions for the elastic field,

\[
\begin{align*}
\mathbf{\Phi} &= [\phi_{11}; \phi_{22}; \phi_{33}; -\phi_{23}; -\phi_{13}; \phi_{12}] \\
\mathbf{R} &= [R_{11}; R_{22}; R_{33}; -R_{23}; -R_{13}; R_{12}] \\
\mathbf{\bar{R}} &= [R'_{11}; R'_{22}; R'_{33}; R'_{23}; R'_{13}; R'_{12}]
\end{align*}
\]

where the two bars denote double derivatives. Note that the last two vectors have different signs for the fourth and fifth members. After lengthy derivation, one can write the displacements as,

\[
u_i = -\frac{1}{8\pi H} \int_0 \left( \mathbf{U}_i - \mathbf{\bar{R}}_i - 2\mathbf{R}_i - 2x_3 \mathbf{\bar{R}}_{3i} + 2x_3^2 \mathbf{\bar{\Phi}}_i \right) \left| \varepsilon \right| d\mathbf{x}
\]

where each \( U_i \) is the summation of three vectors,

\[
\begin{align*}
\mathbf{U}_1 &= 2 \left( \begin{array}{c}
(2 - v)(\phi'_1 + \phi) \\
v(\phi'_1 + \phi) \\

\phi'_1 + \phi \\

\phi' + \phi \\
0 \\

(2 - v)(\phi'_1 + \phi)
\end{array} \right)^T + 4D \left( \begin{array}{c}
v\phi'_1 + H\theta_{111} \\
v\phi'_1 + H\theta_{221} \\
\phi'_1 \\
0 \\
0 \\
H\theta_{121}
\end{array} \right)^T + 4x_3 \left( \begin{array}{c}
v\phi'_{31} \\
v\phi'_{31} \\
(2 - v)\phi'_{31} \\
-H\phi_{21} \\
-H\phi_{11} \\
0
\end{array} \right)^T \\
\mathbf{U}_2 &= 2 \left( \begin{array}{c}
v(\phi'_2 + \phi) \\

\phi'_2 + \phi \\

\phi'_2 + \phi \\

\phi'_2 + \phi \\
0 \\

(2 - v)(\phi'_2 + \phi)
\end{array} \right)^T + 4D \left( \begin{array}{c}
v\phi'_2 + H\theta_{112} \\
v\phi'_2 + H\theta_{222} \\
\phi'_2 \\
0 \\
0 \\
H\theta_{122}
\end{array} \right)^T + 4x_3 \left( \begin{array}{c}
v\phi'_{32} \\
v\phi'_{32} \\
(2 - v)\phi'_{32} \\
-H\phi_{22} \\
-H\phi_{12} \\
0
\end{array} \right)^T \\
\mathbf{U}_3 &= 2 \left( \begin{array}{c}
v(\phi'_3 + \phi) \\

(\phi'_3 + \phi) \\

(\phi'_3 + \phi) \\

(\phi'_3 + \phi) \\
0 \\

(2 - v)(\phi'_3 + \phi)
\end{array} \right)^T + 4D \left( \begin{array}{c}
2R_{113} - 8v\phi_{33} + 4HD\phi_{11} \\
2R_{223} - 8v\phi_{33} + 4HD\phi_{22} \\
-4D\phi_{33} \\
0 \\
0 \\
4HD\phi_{12}
\end{array} \right)^T + 4x_3 \left( \begin{array}{c}
\nu\phi'_{33} - D\phi'_{11} \\
\nu\phi'_{33} - D\phi'_{22} \\
(1 + v)\phi_{33} \\
-v\phi_{23} \\
-v\phi_{13} \\
-D\phi_{12}
\end{array} \right)^T
\end{align*}
\]
These expressions in Eq. (9) are arranged in the form to highlight the following points: (1) there are four categories of terms inside the kernel. The first one has a multiplier of $x_i^2$, the second one has a multiplier of $x_j$, the third one has no $x_3$ outside of the integral nor a superscript $I$, and the last one has no $x_3$ outside of the integral but with a superscript $I$; (2) the last category represents the solution for inclusion problems in an infinite space that is identical to those in Eq. (6) of Liu and Wang (2005), and is written in the matrix form in Appendix A.1. The third category in the half-space solution is a companion infinite-space solution where signs denote its relations to the image inclusion introduced to make the surface shear traction free. This approach agrees with the method that Chiu used to solve half-space inclusion problems (Chiu, 1978); (3) all terms in vectors $U$, and later in $\Theta$, are shown in an aligned column for clarity. There is, e.g. in $U_m$, some flexibility for combining terms of the second half-space and terms of the correction part to simplify the expressions. It is easy to identify that the integration in Eq. (9) with respect to $x_i$ and $x_j$ are two-dimensional convolutions. However, due to the presence of $x_3$ inside the integral kernel, in its entirety the integration in Eq. (9) with respect to $x_3$ is neither a convolution nor a correlation. But one can divide the integral into four integrals with terms from the four categories and take $x_3$ out of two of the integrals. The integration in Eq. (9) with respect to $x_3$ is a correlation for the first three categories and a convolution for the last category. These arrangement and observation will be applicable to the stress expressed discussed below.

When $\varepsilon$ can be analytically expressed over the domain of $\Omega$, one can determine the displacements by finding the integrals of the following derivatives,

$$\phi_i, \phi_j, \phi_{ik}, R_{ijk}, \psi_{11}, \psi_{12}, \theta_{111}, \theta_{122}, \theta_{112}, \theta_{222}$$

### 2.3. Stress field outside of $\Omega$

The stress field outside of $\Omega$ consists of the following components,

$$\sigma_{ij}^{\text{out}} = -\frac{H}{4\pi R} \int_{\Omega} \left( \Theta_{ij} - \tilde{R}_{ij} - 2x_j \tilde{R}_{3ij} + 2x_j \tilde{R}_{3ij} \right) \varepsilon^i dx^j$$

where

$$\Theta_{11} = 4\tilde{R}_{33} - 2D\tilde{R}_{11} + 2 \left( \begin{array}{cccc}
0 & \psi_{311} - \psi_{123} & \psi_{323} + 2\psi_{311} & -\psi_{222} - \psi_{211} \\
\psi_{23} - \psi_{23} & \phi_{23} - \phi_{33} & \phi_{23} - \phi_{33} & 0 \\
\phi_{12} - \phi_{21} & \phi_{12} - \phi_{21} & \phi_{12} - \phi_{21} & 0 \\
\phi_{11} + \phi_{11} & \phi_{11} + \phi_{11} & \phi_{11} + \phi_{11} & 0
\end{array} \right)^T + 4x_3 \left( \begin{array}{c}
-\psi_{222} - \psi_{211} \\
\phi_{23} - \phi_{23} \\
\phi_{12} - \phi_{21} \\
\phi_{11} + \phi_{11}
\end{array} \right)$$

$$\Theta_{22} = 4\tilde{R}_{33} - 2D\tilde{R}_{22} + 2 \left( \begin{array}{cccc}
0 & \psi_{311} - \psi_{123} & \psi_{323} + 2\psi_{311} & -\psi_{222} - \psi_{211} \\
\psi_{23} - \psi_{23} & \phi_{23} - \phi_{33} & \phi_{23} - \phi_{33} & 0 \\
\phi_{12} - \phi_{21} & \phi_{12} - \phi_{21} & \phi_{12} - \phi_{21} & 0 \\
\phi_{11} + \phi_{11} & \phi_{11} + \phi_{11} & \phi_{11} + \phi_{11} & 0
\end{array} \right)^T + 4x_3 \left( \begin{array}{c}
-\psi_{222} - \psi_{211} \\
\phi_{23} - \phi_{23} \\
\phi_{12} - \phi_{21} \\
\phi_{11} + \phi_{11}
\end{array} \right)$$

$$\Theta_{12} = 4\tilde{R}_{33} - 2D\tilde{R}_{12} + 2 \left( \begin{array}{cccc}
0 & \psi_{311} - \psi_{123} & \psi_{323} + 2\psi_{311} & -\psi_{222} - \psi_{211} \\
\psi_{23} - \psi_{23} & \phi_{23} - \phi_{33} & \phi_{23} - \phi_{33} & 0 \\
\phi_{12} - \phi_{21} & \phi_{12} - \phi_{21} & \phi_{12} - \phi_{21} & 0 \\
\phi_{11} + \phi_{11} & \phi_{11} + \phi_{11} & \phi_{11} + \phi_{11} & 0
\end{array} \right)^T + 4x_3 \left( \begin{array}{c}
-\psi_{222} - \psi_{211} \\
\phi_{23} - \phi_{23} \\
\phi_{12} - \phi_{21} \\
\phi_{11} + \phi_{11}
\end{array} \right)$$

(12a), (12b)
2.4. Stress field inside

In order to facilitate code development, Appendix A.2 gives the stress field in a unified form.

When \([e]\) can be analytically expressed over \(\Omega\), one can determine these stress components by finding the integrals of the following derivatives.

\[
\phi_{ij} \phi_{ikl} \delta_{ijkl} R_{ijkl} \phi_{1111} \phi_{1112} \phi_{1122} \phi_{2222}
\]

In order to facilitate code development, Appendix A.2 gives the stress field in a unified form.

2.4. Stress field inside \(\Omega\)

According to the potential theory, one can obtain

\[
\phi_{ij} = -4\pi \delta(x - x')
\]

where \(\delta()\) is the Dirac delta function. Eshelby (1957) gave the following integral

\[
\left( \int_{\Omega} \phi_{ij} dx \right)_{\bar{\omega}} = \begin{cases} -4\pi, & x \in \Omega \\ 0, & x \notin \Omega \end{cases}
\]
which can be readily verified, based on Eq. (13). Furthermore, it is also true that
\[
\left( \int_{\Omega} e_{ij} \phi' \, d\mathbf{x} \right)_{kk} = \begin{cases} -4\pi e_{ij}(\mathbf{x}), & \mathbf{x} \in \Omega \\ 0, & \mathbf{x} \notin \Omega \end{cases}
\]  

(15)
The term of \( \lambda \delta_{ik} \mu_{ik} \) in Eq. (1b) contains,
\[
\lambda \delta_{ij} \int_{\Omega} 2v \varepsilon_{ik} \phi' \, d\mathbf{x} = -8\pi \lambda \delta_{ij} \varepsilon_{kk}, \quad \mathbf{x} \in \Omega
\]
which can be combined with the last term in Eq. (1c). Therefore, based on Eq. (1c), the stress field inside \( \Omega \) can be written as,
\[
\sigma_{ij}(\mathbf{x}) = \sigma_{ij}^{\text{out}}(\mathbf{x}) - 2\mu e_{ij} - \frac{2\mu}{1 - v} \varepsilon_{kk} \delta_{ij}
\]

(17)
where \( \sigma_{ij}^{\text{out}}(\mathbf{x}) \) means evaluating Eq. (11) at point \( \mathbf{x} \), which is inside \( \Omega \).

2.5. Displacement at the surface

When \( x_3 = 0 \), one can consolidate the potentials related to the infinite space into those without superscript \( I \). The following identities are valid for these two types of potentials, \( R^I \) and \( \phi^I \):

(i) when the total derivative number is odd, e.g., \( R^I_3 = -R_3^I, \phi^I_3 = -\phi_3 \),
(ii) when the total derivative number is even, e.g., \( R^I_33 = R_33^I, \phi^I_33 = \phi_33 \).

Therefore,
\[
u^I = -\frac{1}{2\pi} \int_{\Omega} U^I[e] \, d\mathbf{x}
\]

(18)
where
\[
U^I_1 = \overline{R}^I_1 + \begin{pmatrix} 2(1 + v)\phi_1 + D\theta_{111} \\ 2v\phi_1 + D\theta_{221} \\ \phi_1 \\ 0 \\ -\phi_3 \\ \phi_2 + D\theta_{121} \end{pmatrix}^T
\]

(19a)
\[
U^I_2 = \overline{R}^I_2 + \begin{pmatrix} 2v\phi_2 + D\theta_{112} \\ 2(1 + v)\phi_2 + D\theta_{222} \\ \phi_2 \\ -\phi_3 \\ 0 \\ \phi_1 + D\theta_{122} \end{pmatrix}^T
\]

(19b)
\[
U^I_3 = \overline{R}^I_3 + \begin{pmatrix} -2v\phi_3 + D\psi_{11} \\ -2v\phi_3 + D\psi_{22} \\ -3\phi_3 \\ \phi_2 \\ \phi_1 \end{pmatrix}^T
\]

(19c)

When \( [e] \) can be analytically expressed over the domain of \( \Omega \), one can determine the surface displacements by finding the integrals of the following derivatives,
\[
\phi_{i3}, R_{i3}, \psi_{11}, \psi_{22}, \psi_{12}, \theta_{111}, \theta_{122}, \theta_{112}, \theta_{222}
\]

fewer than those required for displacements anywhere inside the half-space.

In summary, for the first time, all integral kernels are explicitly derived for the entire elastic field inside a half-space subject to arbitrary inclusions.
3. Solution when the region is cuboidal and eigenstrains are uniform

In the formulae for the elastic field, volumetric integrals consist of convolution and correlation. When region $\Omega$ has rectangular boundaries (Fig. 2) and uniform eigenstrains, Chiu (1978) used the method of images to construct a solution of the elastic field inside a half-space. In this section, this problem is solved with the newly derived formulae in Section 2. The center of $\Omega$, denoted as $X$, has coordinates of $(o_1, o_2, o_3)$. When the size of $\Omega$ is $2 \Delta_1 \times 2 \Delta_2 \times 2 \Delta_3$, domain $\Omega$ can be expressed as,

$$X_1 \in [o_1 - \Delta_1, o_1 + \Delta_1]; \ \ X_2 \in [o_2 - \Delta_2, o_2 + \Delta_2]; \ \ X_3 \in [o_3 - \Delta_3, o_3 + \Delta_3]$$

(20)

For the elastic field expressions described in the above Eqs. (9-19), one may use variable substitution of

$$\begin{align*}
\zeta_1 &= x_1 - x_1' \\
\zeta_2 &= x_2 - x_2' \\
\zeta_3 &= x_3 - x_3' \quad \text{in } \mathbb{R} \quad \text{and } \phi' \\
\zeta_3 &= x_2 + x_3' \quad \text{in } \mathbb{R} \quad \text{and its related potentials}
\end{align*}$$

(21)

After variable substitution and change of the integral limits so that the small value is the lower limit and the large value is the upper limit, the elastic field is a summation of several integrals times their corresponding coefficients. The convolution integrals can be written as

$$\int_{x_1 - o_1 - \Delta_1}^{x_1 + o_1 + \Delta_1} \int_{x_2 - o_2 - \Delta_2}^{x_2 + o_2 + \Delta_2} \int_{x_3 - o_3 - \Delta_3}^{x_3 + o_3 + \Delta_3} G(\zeta_1, \zeta_2, \zeta_3) d\zeta$$

(22a)

and the integrals with correlation-convolution combination can be written as,

$$\int_{x_1 - o_1 - \Delta_1}^{x_1 + o_1 + \Delta_1} \int_{x_2 - o_2 - \Delta_2}^{x_2 + o_2 + \Delta_2} \int_{x_3 - o_3 - \Delta_3}^{x_3 + o_3 + \Delta_3} G(\zeta_1, \zeta_2, \zeta_3) d\zeta$$

(22b)

Although Eq. (22a) and (22b) have different integral limits for $d\zeta$, their 3D indefinite integrals are identical, and will be evaluated in this section. Note that proper integral constants will be added or omitted in the indefinite integrals (not unique) in order to reach concise formulae, and these constants do not affect the definite integrals. Also note that the summation convention is not used in this section. Four key integrals are used in the expressions of the elastic field, i.e. Eqs. (9), (11), (17) and (18).

$$\begin{align*}
A_0 &= \int \left[ r - \zeta_3 \ln(r + \zeta_3) \right] d\zeta; \ A_1 = \int \frac{1}{r} d\zeta; \\
A_2 &= \int r d\zeta; \ A_3 = \int \ln(r + \zeta_3) d\zeta
\end{align*}$$

(23)

where $r^2 = \zeta_1^2 + \zeta_2^2 + \zeta_3^2$. One can see that $A_0$, $A_1$, $A_2$, and $A_3$ are related to $\theta$, $\psi$, $R$, and $\psi$, respectively. It is obvious that $A_{1,2,3} = A_k$. However, other derivatives of $A_k$ are necessary for evaluating the elastic field. Define five functions to simplify integral expressions,

$$\begin{align*}
U_k &= \tan^{-1} \frac{\zeta_3^2}{\zeta_k^2}; \ V_k = \frac{1}{r(r + \zeta_k)}; \ W_k = \frac{2r + \zeta_k}{r^2(r + \zeta_k)^2}; \ X_k = \tan^{-1} \frac{\zeta_k}{\zeta_1 + \zeta_2}; \ Y_j = \ln(r + \zeta_3)
\end{align*}$$

(24)

One can find that $A_0$ has the following derivatives, where $k$ and $l$ are different and can only take a value of 1 or 2,

$$\begin{align*}
A_{0,kl} &= \frac{1}{2} \left( \zeta_k Y_3 + \zeta_3 Y_k \right); \ A_{0,kkl} = \frac{1}{2} \zeta_k Y_3 + \zeta_3 Y_k - \zeta_k \zeta_3; \\
A_{0,kl} &= \frac{\zeta_k \zeta_3}{2(r + \zeta_3)^2}; \ A_{0,kkk} = -A_{0,kkl} - U_k; \ A_{0,akk} = \frac{1}{2} Y_3 + \frac{\zeta_3^2}{2(r + \zeta_3)^2} + \frac{\zeta_k}{2(r + \zeta_3)}
\end{align*}$$

(25)

![Fig. 2. Cuboidal inclusion with uniform eigenstrain in an elastic half-space.](image-url)
Since $\xi_1$, $\xi_2$, and $\xi_3$ are interchangeable in kernels of $A_1$ and $A_2$, derivatives for $A_1$ and $A_2$ can be concisely written with indices. In the following equations, indices are different ($j \neq k \neq l$) and each has a value of 1, 2, or 3.

\begin{align}
A_{1,j} &= \xi_j Y_i + \xi_i Y_j - \xi_k U_k; \quad A_{1,k} = -Y_j; \quad A_{1,l} = -U_k; \\
A_{1,kl} &= \xi_l V_j; \quad A_{1,kk} = -\xi_j V_l - \xi_i V_j; \quad A_{1,123k} = -\xi_k/R^3; \\
A_{2,kl} &= \xi_l W_j; \quad A_{1,123k,kk} = \xi_k V_j + \xi_k W_j; \quad A_{1,123k,kl} = V_j - \xi_k^2 W_j \\
A_{2,kl} &= \xi_l Y_j; \quad A_{2,123k,kk} = \xi_k Y_j - 2\xi_k U_k; \quad A_{1,123k,kl} = \xi_k V_j; \\
A_{2,123k} &= \xi_k\xi_l V_j - \xi_k\xi_i V_j - 2U_k; \quad A_{2,123k,kk} = \xi_k V_j - \xi_k^2 W_j; \\
A_{2,123k,kl} &= 3\xi_k V_j - \xi_k^2 W_j; \quad A_{2,123k,kl} = \xi_k^2 W_j - 3\xi_k V_i + \xi_k^2 W_l - 3\xi_k V_l
\end{align}

(26a)

The term $A_{3,kl}$ is only used in $U_3$ with the following derivatives,

\begin{align}
A_{3,kl} &= -\xi_k Y_i - 2\xi_3 X_i; \quad A_{3,12} = \xi_3 Y_3 - r
\end{align}

(26c)

where $k$ and $l$ are different and can only take the value of 1 or 2.

One can verify that after substituting these integral into surface displacement expressions and enforcing the zero trace of eigenstrain, the results presented in Jacq et al. (2002) and Fullerger and Nelia (2010) are reproduced.

By substituting Eqs. (21-26) into Eqs. (9), (11), (17) and (18), one solves the elastic fields caused by a cuboidal inclusion in an elastic half-space. All the above formulae are given in explicit closed-form involving elementary functions only, and hence can be readily programmed on a personal computer.

4. Numerical implementation of FFT

In order to solve the elastic field caused by an inclusion of any shape and subjected to arbitrarily distributed eigenstrains, a numerical discretization is performed, i.e., subdividing the region of interest (computational domain) into a set of cuboidal elements (Fig. 3), inside each of which there are approximately uniform eigenstrains. Each element has a representative location of interest, whose displacements and stresses need to be determined. In this paper, it is reasonable to assume that the elastic field for all locations should be determined. With the superposition principle, the resultant elastic field is obtained by superposing contributions from each element. As defined by Johnson (1985), the magnitude of the response at a target point caused by a unit source excitation is the influence coefficient (IC). In the present case, the ICs corresponding to the elastic field components resulting from each eigenstrain component must be determined. There are total $3 \times 6$ and $6 \times 6$ combinations for the displacements and stresses, respectively. If the mesh is not uniform and the total element number is $N_e$, calculations of the displacements and stresses requires the total ICs numbers be $18N_e^2$ and $36N_e^2$, respectively. However, for uniform meshes, these two numbers are $18N_e$ and $36N_e$.

The 3D numerical integration is a huge computational burden. Fortunately, computational science and technology has developed the FFT as a ubiquitous and powerful tool. In mechanics, the FFT is generally applied to efficiently deal with convolutions (Liu et al., 2000) and correlations (Liu and Wang, 2005). Elastic fields of an infinite-space inclusion problem contain 3D convolutions due to the arrangement inside $R^3 = (x_1 - x_1)^2 + (x_2 - x_2)^2 + (x_3 - x_3)^2$. The discrete convolution and FFT (DC-FFT) algorithm discussed in (Liu et al., 2000) includes zero padding and wrap around order within the extended domain. With only a double size in each direction, the DC-FFT algorithm avoids any extra numerical error. The detailed algorithm can be found in Liu and Wang (2002), Liu et al. (2000).

Elastic fields of a half-space inclusion problem are expressed in Eqs. (9), (11), (17) and (18). Using the displacement $u_3$ of Eq. (9) as an example, its expression is written as,

\begin{align}
u_3 &= -\frac{1}{8\pi R} \int \left( U_3 - \overline{R}_3 - 2D\overline{R}_3 - 2x_3\overline{R}_3 + 2x_3^2\overline{B}_3 \right) |e|dx'
\end{align}

(27)

where $U_3 = U_3 + x_3 V_3$ and the two vectors ($U_{3a}$ and $U_{3b}$) are introduced for simplicity. One can see that with respect to $x_1$ and $x_3$ directions, this integration is a 2D convolution. Due to the fact that $x_3$ appears alone in front of $U_{3b}$, $\overline{R}_3$, and $\overline{B}_3$, this kernel as its entirety is not a convolution nor a correlation with respect to $x_3$, the depth direction. However, one way to cope with this challenge is to divide the integral into four sub-integrals:

\begin{align}
\frac{1}{8\pi R} \left\{ \int_{\Omega_3} |e|dx' - \int_{\Omega_3} (U_{3a} - \overline{R}_3 - 2D\overline{R}_3)|e|dx' + x_3 \int_{\Omega_3} (2\overline{R}_3 - U_{3b})|e|dx' - 2x_3^2 \int_{\Omega_3} \overline{B}_3|e|dx' \right\}
\end{align}

(28)

where the kernels have four categories as mentioned in Section 2. At this point, there are two types of integrals: (1) The first category, $\overline{R}_3$ , which corresponds to the initial infinite-space, results in a 3D convolution as mentioned in previous paragraph and can be evaluated with the DC-FFT algorithm; (2) the last three integrals has a 2D convolution with respect to $x_1$ and $x_2$ and 1D correlation of $x_3$. Therefore one can use the combination of the DC-FFT algorithm and the discrete correlation and fast
Fourier transform (DCR-FFT) algorithm as explained in Liu and Wang (2005) to evaluate them. Both types require 3D FFT. After four sub-integrals are determined for all interested locations, a final assembly following Eq. (28) gives the values of $u_3$.

5. Results and discussions

The present analytical solution for the full ICs matrix ($6 \times 6$) due to a cuboidal inclusion in an elastic half-space is numerically compared to Chiu’s solution (Chiu, 1978) as corrected by Jacq et al. (2002). Both solutions were implemented in a double precision computer code programmed in the FORTRAN language. The computations were performed on a personal computer with a 2.67 GHz i7 CPU and 3G memory. Extensive numerical comparisons demonstrated that the expressions from Sections 2 and 3 give almost identical (up to the round-off error) results as Chiu’s solution, but roughly consuming only one quarter of the computation time, which will be detailed later. The major factor contributing to this improvement is that the present analytical solution reported here is explicit and the corresponding calculations are performed in an efficient and straightforward manner; while the indirect manipulations of the recursive relations in Chiu’s approach (1978) might significantly inhibit computational efficiency.

A 3D problem usually has a huge total element number, $N_e$, resulting in severe CPU time penalty over the ICs evaluation. For instance, even a very coarse computational mesh has 4096 elements. If FFT algorithm is not employed after numerical discretization, the resultant elastic fields have to be obtained by directly superposing the contribution of each element. In the case that the numerical results need to be reported at one representative location for each element, the computational time cost by ICs evaluation is proportional to $N_e^2$, as demonstrated in Fig. 4, where the CPU time (the $y$-axis) is represented on a logarithmic scale. For a total number of 4096 elements, evaluating ICs by using Chiu’s solution consumes 7689 s (>2 h) of CPU time. The present analytical solution requires 1898 s. It is concluded from this benchmark that the present analytical solution gives an improvement of about 4 times the computational efficiency over the previous one (Chiu, 1978; Jacq et al., 2002).

Furthermore, the finite element method (FEM) with ABAQUS™ is used to benchmark the analytical solution. A half-space is modeled with dimensions of $40 \times 40 \times 20$ mm$^3$ and the 8-node thermally coupled brick (type 3D3T) elements. The con-

![Fig. 3. Discretization of the computational domain into $n_1 \times n_2 \times n_3$ cuboidal elements.](image1)

![Fig. 4. Comparisons of the computational efficiency between the present analytical solution with Chiu’s solution for evaluating ICs.](image2)
figuration and computational parameter are listed in Table 1. In the FEA model, eigenstrain is prescribed by specifying an equivalent anisotropic thermoelastic expansion of \( e_{ij} = a_{ij} \Delta \) in the cuboidal region (Fig. 5(a)), where \( a_{ij} \) is the linear thermal expansion coefficient, and \( \Delta \) is a unit temperature change. FEM results are output along a vertical reference target line at \( x_1 = 0.0 \) and \( x_2 = 0.5 \) mm, parallel to the \( x_3 \) axis (Fig. 5(a)), and compared with the analytical results derived from the present solutions. Fig. 5(b)–(d) demonstrate satisfactory agreement for both displacements and stresses. All these stresses results (Fig. 5(c) and (d)) agree with Chiu’s solution. As can be seen from the analytical solutions in Fig. 5(c) and (d), the stresses \( \sigma_{11}, \sigma_{22} \) and \( \sigma_{12} \) are discontinuous across the boundary of the cuboidal domain, while the other stresses are continuous. The coarse FEA mesh is not able to capture this discontinuity well.

When FFT algorithms are employed in the computation, uniform meshes are enforced, which lead to orders of magnitude savings in ICs evaluation. In the existing method implementing 2D FFT (Jacq et al., 2002), the computational domain is decomposed into individual horizontal layers. The contributions of one layer to another only involve 2D discrete convolutions, which are calculated by implementing 2D FFT. The resultant elastic fields are consequently obtained after the 2D FFTs are performed layer by layer. In the present 3D FFT computations, the ICs are categorized into 4 different groups, which are evaluated separately. Nevertheless, the costs of ICs evaluation in 3D FFT are the most economical, as discussed in Section 4. For a computational domain, the 2D FFT and 3D FFT algorithms take about 18 and 7 s, respectively, for computing the ICs, which all tend to be orders of magnitude savings as compared to the direct method discussed earlier (>30 min). However, as the meshes being densified in the computational domain, the ICs evaluation in the 3D FFT consumes only a fraction of CPU time as required by that of the 2D FFT (Fig. 6). It is also noted that for a discretization, the computation using 2D FFT was aborted due to stack overflow, and hence the exact computational time (which is estimated to be 20 h) is not available. Note that this phenomenon also occurs even when single precision is adopted in the computation, as reported in more details in Zhou et al. (2009).

### Table 1

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cubic size, ( 2a \times 2b \times 2c ) (mm)</td>
<td>( 2.25 \times 2.25 \times 2.25 )</td>
</tr>
<tr>
<td>Cubic center, ( (0,0,0) ), (mm)</td>
<td>( (0,0,2) )</td>
</tr>
<tr>
<td>Poisson’s ratio, ( v )</td>
<td>0.3</td>
</tr>
<tr>
<td>Uniform eigenstrains in cubic region</td>
<td>( [e] = 10^{-3} \times [1;1;1;1;1;1]^T )</td>
</tr>
</tbody>
</table>
The overall computation time mainly consists of evaluation of ICs and the performance of the corresponding FFT algorithms. The CPU time required for performing 2D FFT or 3D FFT is reported in Table 2. It is seen again that the present 3D FFT is more efficient in conducting the FFT manipulations as compared with the 2D algorithm, although the savings are not as substantial as in the ICs cases.

When performing elastic–plastic analysis of contact problems, iteration methods are often needed to simulate contact or lubrication performance. Normally the ICs are calculated once and stored in memory. However, the iteration demands repeat evaluation of elastic fields, which adds a huge computational burden. For a computational domain consisting of \( N_e = N_1 \times N_2 \times N_3 \) cuboidal elements, the calculation of a component of the elastic field at a node using the direct multiplication method requires \( N_e^2 \) multiplications. If one uses the 2D FFT, the multiplications are on the order of \( N_1N_2\log_2(N_1N_2) \), i.e., the time saving is folds. The usage of the 3D FFT further reduces the multiplications to the order of \( N_1N_2N_3\log_2(N_1N_2N_3) \), and the time saving is \( N_1N_2N_3\log_2(N_1N_2N_3) \) folds.

By treating plastic strains as eigenstrains (Jacq et al., 2002), the present cuboidal inclusion solution can be utilized as a valuable tool in developing elasto-plastic contact analysis. An illustrative plastic contact model (Fig. 7) is considered to demonstrate this development. In the plastic contact simulation, a rigid ball is brought in contact against an elasto-plastic ball. The lower ball has a fixed radius, \( R \), and the time saving is \( \lambda \). The computational domain is set at the lower deformable ball, and is uniformly discretized into a system of cuboidal elements (Fig. 7).

A general constitutive model for small stain, rate-independent elasto-plasticity (Belytschko et al., 2000), is adopted in the present study:

\[
\begin{align*}
\sigma &= C : (\dot{e} - \dot{\varepsilon}^p), \quad \dot{\varepsilon}^p = \frac{\partial f(\sigma, \mathbf{q})}{\partial \sigma} \dot{\lambda}, \quad \mathbf{q} = \dot{\lambda} \cdot \mathbf{h(\sigma, \mathbf{q})} \\
\dot{f} &= f_{\sigma} : \sigma + f_{\mathbf{q}} : \mathbf{q}, \quad \dot{\lambda} \geq 0, \quad f \leq 0, \quad \dot{f} = 0
\end{align*}
\]

where \( C \) is the elastic modulus tensor; the symbol “:” denotes tensor contraction between a fourth-rank tensor and a second-rank tensor; \( \dot{\varepsilon}^p \) is a scalar variable representing the rate of plastic flow; vector \( \mathbf{q} \) is internal variables describing the parametric evolution during the process of the plastic flow; \( \mathbf{h} \) denotes the evolution function; and \( f \) is the yield function. From a given set of \( (\dot{\varepsilon}_t, \dot{\varepsilon}^p_t, \mathbf{q}_t) \) at time \( t \) and strain increment \( \Delta \varepsilon_0 = \Delta \dot{\varepsilon} \), the numerical constitutive integration algorithm is conducted to compute \( (\dot{\varepsilon}^p_{t+1}, \mathbf{q}_{t+1}) \) at time \( t + 1 \), by enforcing the loading–unloading conditions \( (\dot{\lambda} \geq 0, f \leq 0, \dot{f} = 0) \), and then \( \sigma_{t+1} \) is updated (cf. Simo and Hughes, 1998, p. 33). The return mapping algorithm for rate-independent plasticity (Simo and Taylor, 1985; Simo and Hughes, 1998) is used to solve the nonlinear Eq. (29). It should be noted that the stress in Eq. (29) can be expressed as follows.
where $r_{ij}^{0}$ is the elastic stress due to contact pressure and can be obtained by using the related influence coefficients and a FFT method (Liu and Wang, 2002). The remaining part on the right hand side of Eq. (8), i.e. $r_{ij}^{r}$, is the residual stresses, which are the stress components induced by the plastic strains and can be calculated by using Eqs. 11 and 17 developed in this work. Performing elasto-plastic contact analysis follows a standard procedure and the interested readers are referred to the flow-charts in Jacq et al. (2002) and Wang et al. (2010) for the detailed numerical scheme.

Typical computational results are plotted in Figs. 8–10, where all the stress results and geometrical parameters are non-dimensionalized by the Young’s modulus and the radius of the lower ball, respectively. Numerical results show that the normalized maximum residual von Mises stress, $\sigma_v^{res}/E$, decreases as the increase of the radius of the rigid ball (Fig. 8). The results converge to the case of a rigid flat surface in contact with an elasto-plastic sphere, when the ratio of $R_1/R_2$ is greater than 30. When $R_1/R_2 \leq 12$, the maximum residual von Mises stress occurs at the surface of the lower ball. However, it experiences a sharp transition and is located inside the subsurface when $R_1/R_2 = 15$. The distribution of the non-dimensionalized residual von Mises stress along the plane $y = 0$ is shown in the contour plot (Fig. 9) for $R_1/R_2 = 4$. To test the dependence of the solution on the resolution of discretization, four different types of meshes are examined and the contour plots of the residual von Mises stress distribution along the plane, $x_2 = 0$, are compared in Fig. 9. The coarsest mesh, $16 \times 16 \times 16$, seems to be inadequate to capture the location of the maximum residual von Mises stress. As the meshes become finer than $32 \times 32 \times 32$, the numerical results are convergent and each show similar distributions. When the computational domain is discretized into $128 \times 128 \times 128$ elements, the maximum residual von Mises stress is found to be located at $(0.222R_2, 0, 0)$, which is on the surface near the contact edge (Fig. 9(d)). Picking this as a representative point, it is interesting to examine the stress variations along the depth direction. Numerical results are compared for 3 different meshes (Fig. 10). Benchmark results show that the solution derived from coarsest mesh, $16 \times 16 \times 16$, has a larger deviation from those of $128 \times 128 \times 128$, particularly for the depth less than $0.02R_2$. However, the $32 \times 32 \times 32$ mesh can roughly offer a reasonable prediction for the stress variation versus the depth.
The inclusion problem of an elastic half-space is investigated by both analytical and theoretical means. Based on the Galerkin vectors, the integral kernels for the elastic fields due to arbitrary eigenstrains are obtained and for the first time explicitly expressed. In its entirety, each component of the elastic field is an integration with a 2D convolution within every plane parallel to the free surface, but not a convolution nor a correlation in the depth direction. However, after dividing each kernel into four groups, a sub-integration of each group includes 3D convolution/correlation, which can be numerically implemented with FFT-based algorithms to achieve efficient and accurate numerical computations. The analytical solution corresponding to a cuboidal inclusion with uniform eigenstrain is obtained in an explicit closed-form and is utilized to derive...

6. Conclusions

The inclusion problem of an elastic half-space is investigated by both analytical and theoretical means. Based on the Galerkin vectors, the integral kernels for the elastic fields due to arbitrary eigenstrains are obtained and for the first time explicitly expressed. In its entirety, each component of the elastic field is an integration with a 2D convolution within every plane parallel to the free surface, but not a convolution nor a correlation in the depth direction. However, after dividing each kernel into four groups, a sub-integration of each group includes 3D convolution/correlation, which can be numerically implemented with FFT-based algorithms to achieve efficient and accurate numerical computations. The analytical solution corresponding to a cuboidal inclusion with uniform eigenstrain is obtained in an explicit closed-form and is utilized to derive...

Fig. 9. Contour view of the distribution of the normalized residual von Mises stress along the plane $y = 0$, for $R_1/R_2 = 4$. Numerical computations were conducted on 4 different meshes: (a) $16 \times 16 \times 16$; (b) $32 \times 32 \times 32$; (c) $64 \times 64 \times 64$; and (d) $128 \times 128 \times 128$.

Fig. 10. Stress variations with depth along the vertical line passing the maximum residual von Mises stress located at $(0.22 R_2, 0, 0)$. Numerical results are compared for 3 different meshes: $16 \times 16 \times 16$; $32 \times 32 \times 32$; and $128 \times 128 \times 128$. 
the expression of influence coefficients (ICs). The evaluation of the ICs by utilizing the present analytical solutions tends to be much more straightforward and efficient than in the previously developed solutions. All formulae and algorithms presented in this study are verified through numerical benchmarks and computation time is discussed. The present solution and numerical implementation can be used as building blocks for analyzing arbitrarily distributed thermal strains, plastic strains, material inhomogeneities and others.

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Appendix A

A.1. Elastic field of an infinite-space in matrix form

The displacements of an infinite-space can be expressed as,

$$u_i = \frac{-1}{8\pi H} \int_{\Omega} (2\mathbf{u} - \mathbf{R}) \epsilon \, d\mathbf{x}$$

where

$$\mathbf{U} = \begin{bmatrix} (2-v)\phi_1^1 & \nu\phi_2^1 & \nu\phi_3^1 & 0 & H\phi_1^1 & H\phi_1^2 \\ \nu\phi_2^1 & (2-v)\phi_2^2 & \nu\phi_3^2 & H\phi_1^2 & 0 & H\phi_1^2 \\ \nu\phi_3^1 & \nu\phi_3^2 & (2-v)\phi_3^3 & H\phi_1^3 & H\phi_1^3 & 0 \end{bmatrix}$$

One can see that there are patterns inside this matrix: If it is divided into two symmetric sub-matrices, (1) diagonal members in each sub-matrix have same coefficients multiplying the potentials; and (2) the off-diagonal entries have the same coefficients multiplying the potentials.

The stresses of an infinite-space can be expressed as,

$$\sigma_{ij}^{\text{out}} = \frac{-\mu}{4\pi H} \int_{\Omega} (\Theta_{ij} - \mathbf{R}_{ij}) \epsilon \, d\mathbf{x}$$

where

$$\Theta = \begin{bmatrix} 4\phi_{1,1}^1 & 2\nu(\phi_{1,1}^1 + \phi_{2,2}^1) & 2\nu(\phi_{1,1}^1 + \phi_{3,3}^1) & 2\nu\phi_{2,3}^1 & 2\phi_{1,3}^1 & 2\phi_{1,2}^1 \\ 2\nu(\phi_{1,1}^1 + \phi_{3,3}^1) & 4\phi_{2,2}^2 & 2\nu(\phi_{2,2}^2 + \phi_{3,3}^3) & 2\nu\phi_{2,3}^2 & 2\phi_{2,3}^3 & 2\phi_{2,2}^3 \\ 2\nu(\phi_{1,1}^1 + \phi_{3,3}^1) & 2\nu(\phi_{1,1}^2 + \phi_{2,2}^3) & 4\phi_{3,3}^2 & 2\nu\phi_{2,3}^3 & 2\phi_{2,3}^3 & 2\phi_{2,2}^3 \\ 2\phi_{1,1}^3 & 2\nu\phi_{1,3}^1 & 2\phi_{1,3}^1 & H(\phi_{1,2}^1 + \phi_{1,3}^1) & H\phi_{1,3}^1 & H\phi_{1,2}^1 \\ 2\phi_{2,1}^2 & 2\nu\phi_{2,3}^1 & 2\phi_{2,3}^2 & H\phi_{2,3}^2 & H(\phi_{1,1}^2 + \phi_{2,2}^1) & H\phi_{2,2}^2 \\ 2\phi_{3,1}^3 & 2\nu\phi_{3,3}^1 & 2\phi_{3,3}^3 & H\phi_{3,3}^3 & H\phi_{3,3}^3 & H(\phi_{1,1}^2 - \phi_{2,2}^1) \end{bmatrix}$$

One can also see that there are patterns inside this matrix: If it is divided into four sub-matrices, (1) the upper right and lower left sub-matrices are transposes of each other; (2) diagonal members in each sub-matrices have same coefficients multiplying the potentials; and (3) in each sub-matrices, the off-diagonal entries have the same coefficients multiplying the potentials.

A.2. Unified form for the elastic field of a half-space

The elastic field of a half-space is expressed in a unified form for easy implementation. Here $\kappa = 3 - 4\nu$, $\mathbf{U}$ and $\Theta$ (with underscores) are denoted for distinction.

$$u_i = \frac{-1}{8\pi H} \int_{\Omega} \mathbf{U}_i \epsilon \, d\mathbf{x}$$
where

\[
\begin{align*}
U_1 &= \begin{pmatrix} 2(2 - v)(\phi_{1}^{i} + \phi_{1}) - R_{111}^{i} - \kappa R_{111} + 4HD\phi_{1} + 4v\phi_{3}^{i} - 2x_{3}R_{111} + 2x_{3}\phi_{1}^{i} \\
2v(\phi_{1}^{i} + \phi_{1}) - R_{221}^{i} - \kappa R_{221} + 4HD(\phi_{1} + 4v\phi_{3}^{i} - 2x_{3}R_{221} + 2x_{3}\phi_{1}^{i}) \\
2v(\phi_{1}^{i} + \phi_{1}) - R_{331}^{i} - \kappa R_{331} + 4D\phi_{1}^{i} + (2 - v)x_{3}\phi_{3}^{i} - 2x_{3}R_{331} + 2x_{3}\phi_{3}^{i} \\
- R_{231}^{i} + \kappa R_{231} + 4Hx_{3}\phi_{23} + 2x_{3}R_{2331} - 2x_{3}\phi_{231} \\
2(\phi_{3}^{i} - \phi_{3}) - R_{111}^{i} + \kappa R_{111} - 4Hx_{3}\phi_{1}^{i} + 2x_{3}R_{1231} - 2x_{3}\phi_{1}^{i} \\
2(\phi_{1}^{i} + \phi_{1}) - R_{112}^{i} - \kappa R_{112} + 4HD\phi_{1} + 2x_{3}R_{1232} + 2x_{3}\phi_{1}^{i} \\
\end{pmatrix}^T
\end{align*}
\]

\[
U_2 = \begin{pmatrix} 2(2 - v)(\phi_{2}^{i} + \phi_{2}) - R_{112}^{i} - \kappa R_{112} + 4v\phi_{3}^{i} + 4HD\phi_{3}^{i} + 2x_{3}R_{1132} + 2x_{3}\phi_{1}^{i} \\
2v(\phi_{2}^{i} + \phi_{2}) - R_{222}^{i} - \kappa R_{222} + 4v\phi_{3}^{i} + 4HD\phi_{2}^{i} + 2x_{3}R_{2232} + 2x_{3}\phi_{2}^{i} \\
2v(\phi_{2}^{i} + \phi_{2}) - R_{332}^{i} - \kappa R_{332} + 4D\phi_{2}^{i} - (v - 2)x_{3}\phi_{3}^{i} - 2x_{3}R_{3332} + 2x_{3}\phi_{332} \\
2H(\phi_{3}^{i} - \phi_{3}) - R_{232}^{i} + \kappa R_{232} - 4Hx_{3}\phi_{22} + 2x_{3}R_{2332} - 2x_{3}\phi_{232} \\
- R_{112}^{i} + \kappa R_{112} - 4Hx_{3}\phi_{12} + 2x_{3}R_{1332} - 2x_{3}\phi_{132} \\
2(\phi_{1}^{i} + \phi_{1}) - R_{112}^{i} - \kappa R_{112} + 4HD\phi_{1}^{i} + 2x_{3}R_{1232} + 2x_{3}\phi_{132} \\
\end{pmatrix}^T
\]

\[
U_3 = \begin{pmatrix} 2v(\phi_{3}^{i} + \phi_{3}) - R_{113}^{i} - 8vH\phi_{3}^{i} + \kappa R_{113} + 4HD\phi_{11} + 2x_{3}(2v\phi_{3}^{i} - 2D\phi_{11} - R_{1333}) + 2x_{3}\phi_{13}^{i} \\
2v(\phi_{3}^{i} + \phi_{3}) - R_{223}^{i} - 8vH\phi_{3}^{i} + \kappa R_{223} + 4HD\phi_{22} + 2x_{3}(2v\phi_{3}^{i} - 2D\phi_{22} - R_{2233}) + 2x_{3}\phi_{23}^{i} \\
2(2 - v)(\phi_{3}^{i} + \phi_{3}) - R_{333}^{i} - 4D\phi_{3}^{i} + \kappa R_{333} + (1 + v)x_{3}\phi_{33}^{i} - 2x_{3}R_{3333} + 2x_{3}\phi_{333} \\
2H(\phi_{2}^{i} + \phi_{2}) - R_{233}^{i} - \kappa R_{233} - 4v\phi_{23} + 2x_{3}R_{2333} - 2x_{3}\phi_{233} \\
2H(\phi_{1}^{i} + \phi_{1}) - R_{113}^{i} - \kappa R_{113} - 4vx_{3}\phi_{13} + 2x_{3}R_{1333} - 2x_{3}\phi_{133} \\
- R_{113}^{i} + \kappa R_{113} + 4HD\phi_{12} - 4D\phi_{12} + 2x_{3}R_{1233} + 2x_{3}\phi_{123} \\
\end{pmatrix}^T
\]

and

\[
\sigma_{yy} = \frac{-H}{4\pi H} \int_{\Omega} [\Theta_{ij} |\varepsilon| d\mathbf{x}
\]

\[
\Theta_{ij} = \begin{pmatrix} 4(\phi_{11}^{i} + \phi_{11}) - R_{1111}^{i} - \kappa R_{1111} + 4v(1 - 4v)\phi_{11}^{i} - 8v^{2}\phi_{33}^{i} + 4vR_{1113} + 4H\phi_{1111}^{i} \\
+ 2v(\phi_{11}^{i} + \phi_{22}^{i} - \phi_{33}^{i}) - R_{2211}^{i} - \kappa R_{2211} + 4vR_{2233} + 4v\phi_{11}^{i} \\
+ 4H\phi_{2211}^{i} - 2x_{3}R_{22311} + 4v\phi_{33}^{i} + 4vR_{2233} + 4v\phi_{11}^{i} \\
+ 2v(\phi_{11}^{i} + \phi_{33}^{i} - \phi_{22}^{i}) - R_{3311}^{i} - \kappa R_{3311} + 4D\phi_{11}^{i} + 4vR_{3333} \\
- 16v\phi_{33}^{i} - 4v\phi_{33}^{i} + 4v(2 - v)x_{3}\phi_{311} - 2x_{3}R_{33311} + 2x_{3}\phi_{3311} \\
+ 2v(\phi_{23}^{i} + \phi_{23}^{i}) - R_{2311}^{i} + \kappa R_{2311} - 4vR_{2333} + 8v\phi_{23} - 4Hx_{3}\phi_{211} \\
+ 4v\phi_{23} + 2x_{3}R_{23311} - 2x_{3}\phi_{2311} \\
2(\phi_{13}^{i} + \phi_{13}^{i}) - R_{1331}^{i} + \kappa R_{1331} + 4vR_{1333} + 8v\phi_{13} - 4Hx_{3}\phi_{111} \\
+ 4v\phi_{13} + 2x_{3}R_{13311} - 2x_{3}\phi_{1311} \\
2(\phi_{12}^{i} + \phi_{21}^{i}) - R_{1211}^{i} + \kappa R_{1211} + 4vR_{1233} - 8v^{2}\phi_{21} \\
+ 4H\phi_{1211}^{i} - 4v\phi_{1233} + 2x_{3}R_{12311} + 2x_{3}\phi_{1211}^{i} \\
\end{pmatrix}^T
\]
\[ \begin{align*}
\Theta_{22} &= \left( \begin{array}{c}
2v(\phi_{11}' + \phi_{22}' - \phi_{33}') - R_{1122}' - \kappa R_{1122} + 4vR_{1133} + 4v\phi_{22}' \\
+ 4HD\theta_{1122} - 2x_3R_{1133} + 4v_3(\phi_{322}' - \phi_{311}') + 2x_3^2\phi_{1122}'
\end{array} \right)'
\]
\[ \begin{align*}
\Theta_{33} &= \left( \begin{array}{c}
2v(\phi_{11}' + \phi_{33}' + \phi_{22}') - R_{1133}' + R_{1333} - 2x_3(R_{1133} + 2v\phi_{22}) + 2x_3^2\phi_{1133}'
\end{array} \right) '
\]
\[ \begin{align*}
\Theta_{32} &= \left( \begin{array}{c}
2v(\phi_{32}' + \phi_{32}') - R_{1132}' - R_{1332} - 2x_3(R_{1132} + 2v\phi_{22}) + 2x_3^2\phi_{1132}'
\end{array} \right) '
\]
\[ \begin{align*}
\Theta_{31} &= \left( \begin{array}{c}
2v(\phi_{13}' + \phi_{31}') - R_{1131}' - R_{1331} - 2x_3(R_{1131} + 2v\phi_{22}) + 2x_3^2\phi_{1131}'
\end{array} \right) '
\]
\[
\Theta_i = \left(\begin{array}{c}
2(\phi_{12} + \phi_{12}) - R_{1112} - \kappa R_{1112} + 4vD\phi_{12} + 4HD\phi_{1112} - 2\kappa R_{1112} + 4v\phi_{1112} + 2\phi_{1112} \n
2(\phi_{12} + \phi_{12}) - R_{2212} - \kappa R_{2212} + 4vD\phi_{12} + 4HD\phi_{2212} - 2\kappa R_{2212} + 4v\phi_{2212} + 2\phi_{2212} \n
2(\phi_{12} + \phi_{12}) - R_{3312} - \kappa R_{3312} + 4D\phi_{12} + 4(2 - v)\phi_{312} - 2\kappa R_{3312} + 2\phi_{312} \n
H(\phi_{31} + \phi_{31}) - R_{2312} + \kappa R_{2312} - 4H\phi_{312} + 2x_2R_{2312} - 2\phi_{2312} \n
H(\phi_{32} + \phi_{32}) - R_{1312} + \kappa R_{1312} - 4H\phi_{312} + x_2R_{1312} - 2\phi_{312} \n
H(\phi_{31} + \phi_{32} + \phi_{33}) - R_{1212} - \kappa R_{1212} + 4HD\phi_{1212} - 2x_3R_{1212} + 2\phi_{1212}
\end{array}\right)^T.
\]

References


