HW9 Solutions

1 (30 pts)

(4 pts each diagram, 2 pts each equation, 10 pts total per case)
(-1 pt each incorrect/missing max/min values)

\[ f = bv \Rightarrow v(t) = \frac{5}{b} \sin(2\pi t). \]

\[ f = kx \Rightarrow v(t) = x'(t) = \frac{10\pi}{k} \cos(2\pi t). \]

\[ f = ma \Rightarrow v(t) = \int a(t) dt = -\frac{5}{2\pi m} \cos(2\pi t). \]
2 (30 pts)

2-1 (8 pts)

(2 pts) Loop rules (KVL):
\[ V_{R2} + V_{L1} + V_{C1} + V_{R3} = 0 \]

(2 pts) Node equations (KCL):
\[ i_{L1} = i_{R2} = i_{R3} = i_{C1} \]

(2 pts) Constitutive laws:
\[ V_{L1} = L_1i'_{L1}, \quad V_{R2} = R_2i_{R2}, \quad V_{R3} = R_3i_{R3}, \quad V_{C1} = \frac{q}{C_1}. \]

State variables: \( V_{C1}, i_{L1} \)
State equations:
\[ i'_{L1} = \frac{V_{L1}}{L_1} = -\frac{V_{C1} + V_{R2} + V_{R3}}{L_1} = -\frac{V_{C1} + i_{L1}(R_2 + R_3)}{L_1} \]
\[ V_{C1}' = \frac{q'_{C1}}{C_1} = \frac{i_{L1}}{C_1} \]

2-2 (4 pts)

\[ q''_{C1} = i'_{L1} = -\frac{V_{C1} + i_{L1}(R_2 + R_3)}{L_1} = -\frac{1}{L_1C_1}q_{C1} - \frac{R_2 + R_3}{L_1}q'_{C1} \]
\[ \Rightarrow q''_{C1} + \frac{R_2 + R_3}{L_1}q'_{C1} + \frac{1}{L_1C_1}q_{C1} = 0 \]

2-3 (12 pts)

(4 pts)
(2 pts) Force balance equation:
\[ m_4 a_4 = -f_1 - f_2 - f_3 \]

(2 pts) Geometric continuity:
\[ v_1 = v_2 = v_3 = v_4 \]

(2 pts) Constitutive laws:
\[ f_1 = b_1 v_1, \ f_2 = b_2 v_2, \ f_3 = k_3 x_3. \]

(2 pts) State variables: \( v_4, x_3 \)
State equations:
\[ v_4' = a_4 = -\frac{f_1 + f_2 + f_3}{m_4} = -\frac{(b_1 + b_2) v_4 + k_3 x_3}{m_4} \]
\[ x_3' = v_4 \]

2-4 (6 pts)

(2 pts) The differential equation of the electrical system is
\[ q''_{C1} + \frac{R_2 + R_3}{L_1} q'_{C1} + \frac{1}{L_1 C_1} q_{C1} = 0, \]
\[ \Rightarrow r^2 + \frac{R_2 + R_3}{L_1} r + \frac{1}{L_1 C_1} = 0. \]
The roots are
\[ r = -\frac{R_2 + R_3}{2L_1} \pm \sqrt{\frac{1}{4} \left( \frac{R_2 + R_3}{L_1} \right)^2 - \frac{1}{L_1 C_1}}. \]

(2 pts) The differential equation of the mechanical system is
\[ x_3'' = v_4' = -\frac{(b_1 + b_2) v_4 + k_3 x_3}{m_4} = -\frac{b_1 + b_2}{m_4} x_3' - \frac{k_3}{m_4} x_3, \]
\[ \Rightarrow x_3'' + \frac{b_1 + b_2}{m_4} x_3' + \frac{k_3}{m_4} x_3, \]
\[ \Rightarrow r^2 + \frac{b_1 + b_2}{m_4} r + \frac{k_3}{m_4} = 0. \]
The roots are
\[ r = -\frac{b_1 + b_2}{2m_4} \pm \sqrt{\frac{1}{4} \left( \frac{b_1 + b_2}{m_4} \right)^2 - \frac{k_3}{m_4}}. \]

(2 pts) Hence, the two systems are equivalent if \( b_1 = R_2, b_2 = R_3, k_3 = 1/C_3, L_1 = m_4. \)
3 (40 pts)

3-1 (8 pts)

(2 pts) Loop rules (KVL):
\[ V_b - V_L - V_C = 0, \quad V_C - V_R = 0 \]

(2 pts) Node equations (KCL):
\[ i_L = i_C + i_R \]

(2 pts) Constitutive laws:
\[ V_L = L \frac{dq_c}{dt}, \quad V_R = R i_R, \quad V_C = q_c / C. \]

(2 pts)
State variables: \( V_C, i_L \)
State equations:
\[ i'_L = \frac{V_L}{L} = \frac{V_b - V_C}{L} \]
\[ V'_C = \frac{q'_c}{C} = \frac{i_C}{C} = \frac{i_L - i_R}{C} = \frac{i_L}{C} - \frac{V_C}{CR} \]

3-2 (4 pts)
The 2nd order differential equation for \( q_c \) is derived as
\[ q''_c = i'_C - i'_R = \frac{V_b - V_C}{L} - \frac{V'_C}{R} = \frac{V_b}{L} - \frac{1}{LC} q_c - \frac{1}{RC} q'_c \]
\[ \Rightarrow q''_c + \frac{1}{RC} q'_c + \frac{1}{LC} q_c = \frac{V_b}{L} \]

3-3 (4 pts)
The roots of characteristic equation are derived as
\[ r^2 + \frac{1}{RC} r + \frac{1}{LC} = 0 \Rightarrow r = -\frac{1}{2RC} \pm \sqrt{\frac{1}{4(RC)^2} - \frac{1}{LC}}. \]

3-4 (4 pts)
If the system demonstrates oscillatory behaviors, it must follows that
\[ \frac{1}{4(RC)^2} < \frac{1}{LC} \Rightarrow L < 4R^2C. \]
3-5 (4 pts)

(2 pts) Force balance:
\[ ma = F_b - f_2, \quad f_1 = f_2 \]

(2 pts) Geometric continuity:
\[ v_1 + v_2 = v_3 \]

(2 pts) Constitutive laws:
\[ f_1 = kx_1, \quad f_2 = bv_2 \]

(2 pts)
State variables: \(x_1, v_3\)
State equations:
\[
\begin{align*}
x'_1 &= v_1 = v_3 - \frac{f_2}{b} = v_3 - \frac{kx_1}{b} \\
v'_3 &= a = \frac{F_b - f_2}{m} = \frac{F_b - kx_1}{m}
\end{align*}
\]

3-6 (16 pts)

(3 pts) The 2nd order differential equation for \(x_1\) is derived as
\[
\begin{align*}
x''_1 &= v'_3 - \frac{k}{b}x'_1 = \frac{F_b - kx_1}{m} - \frac{k}{b}x_1 \\
\Rightarrow x''_1 + \frac{k}{b}x'_1 + \frac{k}{m}x_1 &= \frac{F_b}{m}
\end{align*}
\]

(3 pts) The roots of characteristic equation are derived as
\[
r^2 + \frac{k}{b}r + \frac{k}{m}r = 0 \Rightarrow r = -\frac{k}{2b} \pm \sqrt{\frac{1}{4} \left( \frac{k}{b} \right)^2 - \frac{k}{m}}.
\]

(2 pts) Hence, to ensure the system has oscillatory behavior, it must satisfy that
\[
\frac{1}{4} \left( \frac{k}{b} \right)^2 < \frac{k}{m} \Rightarrow m < \frac{4b^2}{k}.
\]