HW5 Solutions

1 (10 pts)

1-1 (6 pts)
(2 pts each, 6 pts total)

<table>
<thead>
<tr>
<th></th>
<th>real</th>
<th>imaginary</th>
<th>complex conjugate</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>5</td>
<td>-1</td>
<td>5+j</td>
</tr>
<tr>
<td>b)</td>
<td>3/13</td>
<td>-2/13</td>
<td>3/13 + 2/13j</td>
</tr>
<tr>
<td>c)</td>
<td>0</td>
<td>1</td>
<td>-j</td>
</tr>
</tbody>
</table>

Derivations:

a) 

\[ j^8 - 4j^{10} + j^7 = (j^2)^4 - 4(j^2)^5 + j(j^2)^3 = 1 + 4 - j = 5 - j \]

b) 

\[ \frac{1}{3 + 2j} = \frac{3 - 2j}{(3 + 2j)(3 - 2j)} = \frac{3 - 2j}{9 - 4j^2} = \frac{3 - 2j}{13} \]

c) 

\[ e^{j\pi/2} = \cos(\pi/2) + j \sin(\pi/2) = j \]

1-2 (2 pts)

\[
\frac{x + 1 + j(y - 3)}{5 + 3j} = \frac{[x + 1 + j(y - 3)](5 - 3j)}{(5 + 3j)(5 - 3j)}
\]

\[
= \frac{5x - 3xj + 5 - 3j + (y - 3)(5j + 3)}{34}
\]

\[
= \frac{5x + 3y - 4 + (-3x + 5y - 18)j}{34}
\]

Hence,

\[
\begin{align*}
5x + 3y - 4 &= 34 \\
-3x + 5y - 18 &= 34
\end{align*}
\]

\[
\Rightarrow \begin{cases} x = 1 \\ y = 11 \end{cases}
\]

1-3 (2 pts)
(1 pt each, 2 pts total)

1. Since \( \cos(\pi) = -1 \) and \( \sin(\pi) = 0 \),

\[-1 = \cos(\pi) + j \sin(\pi) = e^{j\pi}.\]

2. Let \( 1 + j\sqrt{3} = A(\cos y + j \sin y) \), then \( A \cos y = 1 \) and \( A \sin y = \sqrt{3} \).

\[
A^2 = A^2(\cos^2 y + \sin^2 y) = 4 \Rightarrow A = 2,
\]

\[
\tan y = \frac{\sin y}{\cos y} = \sqrt{3} \Rightarrow y = \frac{\pi}{3} \text{ or } 60^\circ.
\]

Therefore, \( 1 + j\sqrt{3} = 2e^{j\pi/3} \).
2 (15 pts)

2-1 (8 pts)

(1 pt each, 8 pts total)

1. T \((y'' = 0, y' = -1)\)
2. F \((y'' = 0, y' = 6 \Rightarrow y'' - y' - 6y = -36t - 12)\)
3. F \((y'' = 36e^{6t}, y' = 6e^{6t} \Rightarrow y'' - y' - 6y = 24e^{6t})\)
4. F \((y'' = 4e^{-2t}, y' = -2e^{-2t} \Rightarrow y'' - y' - 6y = 0)\)
5. T \((y'' = 9e^{3t}, y' = -1 + 3e^{3t})\)
6. F \((y'' = 4e^{-2t}, y' = 6 - 2e^{-2t} \Rightarrow y'' - y' - 6y = -36t - 12)\)
7. F \((y'' = 36e^{6t}, y' = 1 + 6e^{6t} \Rightarrow y'' - y' - 6y = 24e^{6t} - 6t - 2)\)
8. T \((y'' = 9e^{3t} + 4e^{-2t}, y' = -1 + 3e^{3t} - 2e^{-2t})\)

The general solution is derived as follows.

Let \(y(t) = Ae^{at} + Be^{-bt} + Ct + D\) with \(a, b > 0\), then

\[y'' = Aa^2e^{at} + Bb^2e^{-bt}, y' = Aae^{at} - Bbe^{-bt} + C.\]

Hence,

\[y'' - y' - 6y = A(a^2 - a - 6)e^{at} + B(b^2 + b - 6)e^{-bt} - 6Ct - (C + 6D) = 6t + 1,\]

which yields

\[\begin{align*}
a^2 - a - 6 &= 0 \\
b^2 + b - 6 &= 0 \\
6C &= -6 \\
C + 6D &= -1
\end{align*}\]

\[\Rightarrow \begin{cases} a = 3 \\ b = 2 \\ C = -1 \\ D = 0 \end{cases}\]

Only a), e) and h) follow the form \(y(t) = Ae^{3t} + Be^{-2t} - t\).

2-2 (4 pts)

(2 pts each, 4 pts total)

1. For \(Y_1(t)\),

\[x'_1 = 2x_2 \Rightarrow 2e^{2t}\cos(4t) - 4e^{2t}\sin(4t) = 2e^{2t}[\cos(4t) - B\sin(4t)] \Rightarrow B = 2.\]

2. For \(Y_2(t)\),

\[x'_1 = 2x_2 \Rightarrow 2e^{2t}\sin(4t) + 4e^{2t}\cos(4t) = 2e^{2t}[\sin(4t) - C\cos(4t)] \Rightarrow C = -2.\]

2-3 (3 pts)

\[X(0) = aY_1(0) + bY_2(0)\]

\[\Rightarrow \begin{cases} a\cos(0) + b\sin(0) = 0 \\ a[\cos(0) - 2\sin(0)] + b[\sin(0) + 2\cos(0)] = 1 \end{cases}\]

\[\Rightarrow \begin{cases} a = 0 \\ a + 2b = 1 \end{cases}\]

\[\Rightarrow \begin{cases} a = 0 \\ b = 1/2 \end{cases}\]
3 (20 pts)

3-1 (6 pts)

(2 pts)

\[ x = Ae^{-j\omega t} + Be^{j\omega t} \]
\[ = A[\cos(\omega t) - j \sin(\omega t)] + B[\cos(\omega t) + j \sin(\omega t)] \]
\[ = (A + B) \cos(\omega t) + (B - A)j \sin(\omega t) \]

(2 pts) For \( x = D \cos(\omega t) \), it follows \( A = B = D/2 \).
(2 pts) For \( x = E \sin(\omega t) \), it follows \( A = -B = Ej/2 \) (or \( -\frac{E}{2j} \)).

3-2 (6 pts)

(2 pts) If only \( x = Ae^{-j\omega t} \) is included, then

\[ x = Ae^{-j\omega t} = A \cos(\omega t) - Aj \sin(\omega t) = D \cos(\omega t). \]

It thus follows \( A = D \) and \( A = 0 \).
(2 pts) Similarly,

\[ x = Ae^{-j\omega t} = A \cos(\omega t) - Aj \sin(\omega t) = E \sin(\omega t) \]

requires \( Aj = -E \) and \( A = 0 \).
(2 pts) Therefore, the real-valued solutions cannot be found other than trivial solutions.

3-3 (8 pts)

(2 pts) Let the real-valued solution be \( x = C \cos(\omega t) + D \sin(\omega t) \).
(2 pts) The problem is to find \( C \) and \( D \) such that

\[ x = Ae^{-j\omega t} + Be^{j\omega t}, \]

where \( A \) and \( B \) are known constants.
(2 pts) It has been shown in 3-1 that

\[ x = Ae^{-j\omega t} + Be^{j\omega t} = (A + B) \cos(\omega t) + (B - A)j \sin(\omega t). \]

(2 pts) Hence, the real-valued solutions are

\[ \left\{ \begin{array}{l} C = A + B \\ D = (B - A)j \end{array} \right. \]

4 (20 pts)

4-1 (4 pts)

(2 pts)

Force balance equation:
\[ m_2 a_2 = -f_1. \]

Geometric continuity equation:
\[ v_1 = v_2. \]

Constitutive law:
\[ f_1 = k_1 x_1. \]
State variables: \( x_1, v_2 \).

State equations:

\[
\begin{align*}
\dot{x}_1 &= v_1 = v_2, \\
\dot{v}_2 &= a_2 = -\frac{k_1}{m_2} x_1 \\
\Rightarrow x''_1 &= -\frac{k_1}{m_2} x_1.
\end{align*}
\]

(2 pts)

Let \( x_1 = Ae^{rt} \), then

\[
x''_1 = Ar^2 e^{rt} = r^2 x_1 \Rightarrow r = \pm j \sqrt{\frac{k_1}{m_2}}.
\]

Therefore, the analytical solution is

\[
x_1(t) = Ae^{\pm j \sqrt{\frac{k_1}{m_2}}}.
\]

Note: \( x_1 = A \cos(t \sqrt{\frac{k_1}{m_2}}) \pm j \sin(t \sqrt{\frac{k_1}{m_2}}) \) and \( x_1 = A_1 \cos(t \sqrt{\frac{k_1}{m_2}}) + A_2 \sin(t \sqrt{\frac{k_1}{m_2}}) \) are also correct answers.

4-2 (16 pts)

1. (2 pts) The real-valued general solution is \( x_1 = A_1 \cos(t \sqrt{\frac{k_1}{m_2}}) + A_2 \sin(t \sqrt{\frac{k_1}{m_2}}) \).

Given \( \sqrt{\frac{k_1}{m_2}} = 2 \), it follows

\[
v_1 = x'_1 = -2A_1 \sin(2t) + 2A_2 \cos(2t).
\]

(2 pts) Plugging in the initial condition: \( x_1(0) = 0.3 \) and \( v_2(0) = 0 \)

\[
\left\{ \begin{array}{c}
A_1 \cos(0) + A_2 \sin(0) = 0.3 \\
-2A_1 \sin(0) + 2A_2 \cos(0) = 0
\end{array} \right. \Rightarrow \left\{ \begin{array}{c}
A_1 = 0.3 \\
A_2 = 0
\end{array} \right.
\]

Accordingly, the analytic solution is

\[
x_1(t) = 0.3 \cos(2t), \quad v_2(t) = -0.6 \sin(2t).
\]

2. (See sample codes and figs at the end.)

(2 pts) Codes of the analytic solution and Euler method.

(2 pts) Plot of spring displacement and mass velocity.

(2 pts, 1 pt each)

(a) Results based on analytic solution and Euler method have similar period of oscillation.

(b) The deviation of Euler method from the analytic solution increases over time.

3. (2 pts) Plugging in the initial condition: \( x_1(0) = 0.3 \) and \( v_2(0) = 0 \)

\[
\left\{ \begin{array}{c}
A_1 \cos(0) + A_2 \sin(0) = 0.3 \\
-2A_1 \sin(0) + 2A_2 \cos(0) = 0.8
\end{array} \right. \Rightarrow \left\{ \begin{array}{c}
A_1 = 0.3 \\
A_2 = 0.4
\end{array} \right.
\]

\[
x_1(t) = 0.3 \cos(2t) + 0.4 \sin(2t), \quad v_2(t) = -0.6 \sin(2t) + 0.8 \cos(2t).
\]

(2 pts) Plots of comparison of two initial conditions (either Euler or analytic)

(2 pts, 1 pt each)

(a) The period of oscillation is the same because it is determined by \( \sqrt{\frac{k_1}{m_2}} \).

(b) The amplitude in the second case is larger because there is a non-zero initial mass velocity that brings extra energy to the system.
5 (20 pts)

5-1 (6 pts)
(0.5 pts each, 3 pts total)

<table>
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<tr>
<th></th>
<th>linear</th>
<th>homogeneous</th>
<th>constant coeff</th>
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<tbody>
<tr>
<td>a)</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>b)</td>
<td>F</td>
<td>T</td>
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<td>c)</td>
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<td>T</td>
<td>F</td>
</tr>
<tr>
<td>d)</td>
<td>T</td>
<td>F</td>
<td>T</td>
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</tbody>
</table>

5-2 (4 pts)
Consider the non-linear differential equation b) \( x'' + 2x' + x^2 = 0 \).
(1 pt) Let \( x_1 \) and \( x_2 \) be two solutions and suppose \( x_3 = x_1 + x_2 \) is also a solution.
(2 pts) \[ x_3'' = (x_1'' + x_2'') = -2(x_1' + x_2') - (x_1^2 + x_2^2) = -2x_3' - (x_1^2 + x_2^2) \]
(1 pt) The equality \( x_1^2 + x_2^2 = x_3^2 = (x_1 + x_2)^2 \) holds if and only if \( x_1x_2 = 0 \). Hence, we cannot superpose solutions for a non-linear differential equation.

5-3 (4 pts)
Consider the linear non-homogeneous differential equation d) \( x'' + 3x = 3 \).
(1 pt) Let \( x_1 \) and \( x_2 \) be two solutions and suppose \( x_3 = x_1 + x_2 \) is also a solution.
(2 pts) \[ x_3'' = (x_1'' + x_2'') = -3(x_1 + x_2) + 6 = -3x_3 + 6 \neq -3x_3 + 3 \]
(1 pt) Hence, we cannot superpose solutions for a non-linear differential equation.

5-4 (4 pts)
Consider the linear homogeneous differential equation c) \( x'' + \exp(t)x = 0 \).
(1 pt) Let \( x_1 \) and \( x_2 \) be two solutions and suppose \( x_3 = x_1 + x_2 \) is also a solution.
(2 pts) \[ x_3'' = (x_1'' + x_2'') = -\exp(t)(x_1 + x_2) = -\exp(t)x_3 \]
(1 pt) Hence, solutions for a non-linear differential equation could be superposed.

6 (15 pts)

6-1 (6 pts)
Taking derivative of 1st state equation yields
\[ x'' = -6x' + 25v' \Rightarrow x'' + 6x' + 25x = 0. \]

6-2 (9 pts)
(5 pts) Let \( x = Ae^{rt} \), then
\[ x'' + 6x' + 25x = Ar^2e^{rt} + 6Are^{rt} + 25Ae^{rt} = 0 \Rightarrow r^2 + 6r + 25 = 0 \Rightarrow r = -3 \pm 4j. \]
(4 pts) Therefore, \( x = Ae^{(-3\pm4j)t} \) and the real-valued solution is
\[ x = e^{-3t}[A_1 \cos(4t) + A_2 \sin(4t)]. \]
% HW5 EA3
% Problem 4-2b

clear; close all
% system constants
k = 1000;
m = 250;
dt = 0.1;

% initial conditions
t(1) = 0;
x(1) = 0.3;
v(1) = 0;

% Euler method
% main loop
for i = 1:100
    a(i) = -k/m*x(i);
v(i+1) = v(i) + a(i)*dt;
x(i+1) = x(i) + v(i)*dt;
t(i+1) = t(i) + dt;
end

% analytic solution
xexact = 0.3*cos(2*t);
vexact = -0.6*sin(2*t);

% plot time vs x
figure
plot(t,x,'-b',t,xexact,'k','linewidth',2)
xlabel('time (s)')
ylabel('spring displacement (m)')
legend('dt=0.1s','analytic','location','southwest')

% plot time vs v
figure
plot(t,v,'-b',t,vexact,'k','linewidth',2)
xlabel('time (s)')
ylabel('mass velocity (m/s)')
legend('dt=0.1s','analytic','location','southwest')
% HW5 EA3
% Problem 4-2c
% using Euler method

clear; close all
% system constants
k = 1000;
m = 250;
dt = 0.1;

% initial conditions 1
t(1) = 0;
x1(1) = 0.3;
v1(1) = 0;
for i = 1:100
    a(i) = -k/m*x1(i);
    v1(i+1) = v1(i) + a(i)*dt;
    x1(i+1) = x1(i) + v1(i)*dt;
    t(i+1) = t(i) + dt;
end

% initial conditions 2
t(1) = 0;
x2(1) = 0.3;
v2(1) = 0.8;
for i = 1:100
    a(i) = -k/m*x2(i);
    v2(i+1) = v2(i) + a(i)*dt;
    x2(i+1) = x2(i) + v2(i)*dt;
end

% plot time vs x
figure
plot(t,x1,'-.b',t,x2,'-k','linewidth',2)
xlabel('time (s)')
ylabel('spring displacement (m)')
legend('v2(0)=0','v2(0)=0.8','location','southwest')

% plot time vs v
figure
plot(t,v1,'-.b',t,v2,'-k','linewidth',2)
xlabel('time (s)')
ylabel('mass velocity (m/s)')
legend('v2(0)=0','v2(0)=0.8','location','southwest')
clear; close all
% system constants
k = 1000;
m = 250;
dt = 0.1;
t = [0:dt:10];
% initial conditions 1
x1 = 0.3*cos(2*t);
v1 = -0.6*sin(2*t);
% initial conditions 2
x2 = 0.3*cos(2*t) + 0.4*sin(2*t);
v2 = -0.6*sin(2*t) + 0.8*cos(2*t);
% plot time vs x
figure
plot(t,x1, '-b', t,x2, '-k', 'linewidth',2)
xlabel('time (s)')
ylabel('spring displacement (m)')
ylim([-1 1])
legend('v2(0)=0', 'v2(0)=0.8', 'location', 'southwest')
% plot time vs v
figure
plot(t,v1, '-b', t,v2, '-k', 'linewidth',2)
xlabel('time (s)')
ylabel('mass velocity (m/s)')
ylim([-1.5 1.5])
legend('v2(0)=0', 'v2(0)=0.8', 'location', 'southwest')
4-2 b) graphs

- Spring displacement (m)
  - Time (s): 0 to 10
  - Displacement values: -1.5 to 2

- Mass velocity (m/s)
  - Time (s): 0 to 10
  - Velocity values: -4 to 4

Legend:
- dt=0.1s
- Analytic
4-2 c) graphs using Euler method
4-2 c) graphs using analytic solution