HW2 Solutions

1 (34 pts)

1-1 (28 pts)

1. (0.5 pt each, 4.5 pts total)

2. (0.5 pt each, 8.5 pts total)

For system A, the constitutive laws give

\[ f_1 = k_1 x_1, \quad f_2 = k_2 x_2, \]
\[ f_3 = b_3 v_3, \quad f_4 = b_4 v_4. \]

The force balance equations are

\[ f_1 = f_2 + f_3, \quad f_2 + f_3 = f_4. \]

The geometric continuity equations are

\[ v_1 + v_2 + v_4 = 0, \quad v_2 = v_3. \]

For system B, the constitutive laws give

\[ f_1 = k_1 x_1, \quad f_2 = k_2 x_2, \]
\[ f_3 = b_3 v_3, \quad f_4 = b_4 v_4. \]

The force balance equations are

\[ f_1 = f_2 + f_3, \quad f_2 + f_3 = f_4, \quad m_5 a_5 = -f_4 \]

The geometric continuity equations are

\[ v_1 + v_2 + v_4 = v_5, \quad v_2 = v_3. \]
The state variables for system A are \( x_1, x_2 \).

EOM are derived as follows.

\[
x'_1 = v_1 = -v_2 - v_4 = -v_5 - v_4
\]
\[
= -\frac{f_3}{b_3} - \frac{f_4}{b_4} = -\frac{f_1 - f_2}{b_3} - \frac{f_1}{b_4}
\]
\[
= -\frac{k_1 x_1 - k_2 x_2}{b_3} - \frac{k_1 x_1}{b_4};
\]
\[
x'_2 = v_2 = v_3
\]
\[
= \frac{f_3}{b_3} = \frac{f_1 - f_2}{b_3}
\]
\[
= \frac{k_1 x_1 - k_2 x_2}{b_3}.
\]

The state variables for system B are \( x_1, x_2, v_5 \).

EOM are derived as follows.

\[
x'_1 = v_1 = v_5 - v_2 - v_4 = v_5 - v_3 - v_4
\]
\[
= v_5 - \frac{f_3}{b_3} - \frac{f_4}{b_4} = v_5 - \frac{f_1 - f_2}{b_3} - \frac{f_1}{b_4}
\]
\[
= v_5 - \frac{k_1 x_1 - k_2 x_2}{b_3} - \frac{k_1 x_1}{b_4};
\]
\[
x'_2 = v_2 = v_3
\]
\[
= \frac{f_3}{b_3} = \frac{f_1 - f_2}{b_3}
\]
\[
= \frac{k_1 x_1 - k_2 x_2}{b_3};
\]
\[
v'_5 = a_5 = -\frac{f_4}{m_5} = -\frac{f_1}{m_5} = -\frac{k_1 x_1}{m_5}.
\]

1-2 (1 pt each, 3 pts total)

For system C, \( m_4 \) is unnecessary because it may be considered as part of the wall.

For system D, damper 3 is unnecessary because it is in parallel with a mass and cannot lengthen or shorten.

For system E, damper 4 is unnecessary because it is free on the right hand side and doesn’t generate any force to the system.
2 (20 pts)

Force balance (FB) equations

\[ m_2 a_2 = f_3 + f_6 - f_1 - f_5, \quad m_4 a_4 = -f_3, \quad f_6 = f_7. \]

Geometric continuity (GC) equations

\[ v_1 = v_5 = v_2, \quad v_2 + v_6 + v_7 = 0, \quad v_2 + v_3 = v_4. \]

1. (1 pt each for a-j, 10 pts total; 2 pt each for k-o, 10 pts total)
   
   (a) T
   
   (b) T
   
   (c) F \( (v_2 + v_6 + v_7 = 0, \quad v_2 + v_3 = v_4 \Rightarrow v_6 + v_7 = v_3 - v_4) \)
   
   (d) T
   
   (e) T
   
   (f) F \( (v_2 + v_3 = v_4) \)
   
   (g) F \( (f_1 + f_5 + m_2 a_2 = f_3 + f_6) \)
   
   (h) T
   
   (i) T
   
   (j) F \( (f_6 = f_7) \)
   
   (k) T
   
   (l) T \( (v_2 = -v_6 - v_7, k_6 x_6 = b_7 v_7, \text{ thus } v_6 > 0, v_7 > 0) \)
   
   (m) F \( (\text{it is possible that } a_2 < 0 \text{ but } v_2 > 0 \Rightarrow v_1 > 0) \)
   
   (n) T \( (a_4 = -f_3/m_4 = -k_3 x_3/m_4 < 0)) \)
   
   (o) F \( (a_2 = (f_3 + f_6 - k_1 x_1 - b_5 v_5)/m_2 \text{ thus the statement does not necessarily hold}) \)

3 (40 pts)

3-1 (4 pts)

(1 pt) The force balance equation is \[ f_1 + f_2 = 0. \]
(1 pt) The geometric continuity equation is

\[ v_1 = v_2. \]

(1 pt each) The constitutive laws give

\[ f_1 = k_1 x_1, \quad f_2 = b_2 v_2. \]

3-2 (4 pts)

The differential equations of motion for state variable \( x_1 \) is

\[ x_1' = v_1 = v_2 = \frac{f_2}{b_2} = -\frac{k_1 x_1}{b_2}. \]

3-3 (6 pts)

(See sample Matlab codes and figs at the end)

(1 pt) Define initial condition and time step.
(1 pt) Update velocity.
(1 pt) Update displacement.
(3 pts) Plot time vs displacement with different \( dt \).
(-0.5 pt each incorrect/missing label/legend)

3-4 (10 pts)

(2 pt) Suppose \( x_1 \) follows the form \( x_1 = Ae^{rt} \), then \( x_1' = Ae^{rt} \).
(2 pt) The initial condition gives \( x_1(0) = 0.03 \), thus \( A = 0.03 \).
(2 pt) The result \( x_1' = -\frac{k_1}{b_2} x_1 \) yields \( r = -k_1/b_2 = -1.5 s^{-1} \).
(2 pt) Therefore, \( x_1(t) = 0.03 x^{-1.5t} \).
(2 pt) The numerical solution is close to the analytical solution when the time step (\( dt \)) is small because the error of Euler method decreases with \( dt \).

3-5 (4 pts)

(1 pt) The force balance equation is

\[ m_3 a_3 = -f_1 - f_2. \]

(1 pt) The geometric continuity equation is

\[ v_1 = v_2 = v_3. \]

(1 pt each) The constitutive laws give

\[ f_1 = k_1 x_1, \quad f_2 = b_2 v_2. \]

3-6 (4 pts)

(2 pts each, 4 pts total) The differential equations of motion for state variables \( x_1, v_3 \) are

\[
\begin{align*}
x_1' &= v_1 = v_3; \\
v_3' &= a_3 = -\frac{f_1 + f_2}{m_3} = -\frac{k_1 x_1 + b_2 v_2}{m_3} = -\frac{k_1 x_1 + b_2 v_3}{m_3}.
\end{align*}
\]
3-7 (8 pts)

(See sample Matlab codes and figs at the end)

(1 pt) Define initial condition and time step.

(1 pt) Update acceleration.

(1 pt) Update velocity.

(1 pt) Update displacement.

(1 pt) Plot time vs spring displacement.

(1 pt) Plot time vs mass velocity.

(-0.5 pt each incorrect/missing label)

(2 pt) The mass velocity first increases as the compressed spring pushes it to the right. The mass is then slowed down by the damper while the spring approaches to its equilibrium.
clear; k = 1500; b = 1000;

% Time step size 1 (dt = 0.5)
dt1 = 0.5;
t1(1) = 0;
x1(1) = 0.03;
for i = 1:10
    v1(i) = (-k/b)*x1(i);
x1(i+1) = x1(i) + v1(i)*dt1;
t1(i+1) = t1(i) + dt1;
end

% Time step size 2 (dt = 0.1)
dt2 = 0.1;
t2(1) = 0;
x2(1) = 0.03;
for i = 1:50
    v2(i) = (-k/b)*x2(i);
x2(i+1) = x2(i) + v2(i)*dt2;
t2(i+1) = t2(i) + dt2;
end
texact = [0:0.01:5];
xexact = 0.03*exp(-1.5*texact);
plot(t1, x1,
'b', t2, x2, 'r', texact, xexact, 'k')
xlabel('Time (s)')
ylabel('Spring displacement (m)')
legend('dt = 0.5s', 'dt = 0.1s', 'Analytic solution')

% Problem 3-7
clear;
k = 2000;
b = 1200;
m = 100;
dt = 0.01;
t(1) = 0;
xs(1) = -0.02;
vm(1) = 0;
for i = 1:300
    vs(i) = vm(i); % from spring EOM (xs')
am(i) = -(k*xs(i) + b*vm(i))/m; % from mass EOM (vm')
xs(i+1) = xs(i) + vs(i)*dt; % Euler method for spring
vm(i+1) = vm(i) + am(i)*dt; % Euler method for mass
t(i+1) = t(i) + dt;
end
subplot(2,1,1)
plot(t, xs)
xlabel('Time (s)')
ylabel('Spring displacement (m)')
subplot(2,1,2)
plot(t, vm)
xlabel('Time (s)')
ylabel('Mass velocity (m/s)')