EA3 – Week 10
Electrical Systems – RLC Circuits, Resonance, Mechanical Analogies

Spring 2018

Quiz 3 this week!
RQ (T/F)

1. An analog volume control knob is probably
   A. a potentiometer
   B. an inductor
   C. a variable resistor

2. With respect to the RLC circuit in the reading:
   D. the circuit has two state variables
   E. the state equations cannot be found analytically

3. The sound signal in your stereo
   F. is just a simple $\sin(\omega t)$ for most music
   G. is a messy mix of different frequencies and amplitudes for most music
   H. has nothing to do with the reading
Outline

• Circuits with inductors, resistors, capacitors
• Tuned LRC circuits
• Resonance
  • $\omega = \sqrt{1/(LC)}$
• Mechanical-Electrical Analogies
• Fun with inductors
What is $i_L$ at $t=0$?

(A) positive  
(B) zero  
(C) negative  
(D) depends on the hand
What is $i_L$ at $t>>0$?

(A) positive
(B) zero
(C) negative
(D) depends on the hand
Inductor-Hand Demo

Remove Battery.
What happens to $i_L$ and $i_H$?

(A) Both drop to zero immediately
(B) $i_L$ remains positive and $i_H = -i_L$
(C) $i_L$ reverses (becomes negative)
Inductor-Hand Demo

\[ i'_L = \frac{V}{L} \]
\[ i_L = t \times \frac{V}{L} \]

If \( i_L = 0.05 \) amp (typical) before battery is removed, we will get \( V_H = 2,500 \) volts after battery is removed...
is analogous to...
is analogous to what electrical system?
is analogous to what mechanical system?
State variables $q_C$ & $i_L$

KVL: $-V_B + V_C + V_L + V_R = 0$
KCL: there’s only one current, $i$

Constitutive laws:
- $V_R = i \cdot R$
- $V_L = L \frac{di}{dt}$
- $V_C = \frac{q}{C}$

Diffeqs:
- $q_C' = i_C = i_L$
- $i_L' = \left( V_B - \frac{q}{C} - i_L \cdot R \right) / L$
State variables $q_{C2}$ & $i_{L1}$

KVL: $V_{L1} + V_{C2} + V_{R3} = V_{in}$

KCL: there’s only one current, $i$

Constitutive laws:

$V_{R3} = i \cdot R3$
$V_{L1} = L1 \frac{di}{dt}$
$V_{C2} = \frac{q_2}{C2}$
Sinusoidal input $V_{in} = V_0 \sin \omega_0 t$

Diffeqs:

\[ q_{C2}' = i \]

\[ i' = \frac{1}{L1} \left( V_0 \sin \omega_0 t - V_R3 - \frac{q_{C2}}{C2} \right) \]

\[ = \frac{1}{L1} \left( V_0 \sin \omega_0 t - i \cdot R3 - \frac{q_{C2}}{C2} \right) \]
\[ q'_C = i \]
\[ i' = \frac{1}{L} (V_0 \sin \omega_0 t - V_R - V_C) \]
\[ = \frac{1}{L} (V_0 \sin \omega_0 t - i R - V_C) \]
\[ i'' = (\omega V_0 / L) \cos \omega_0 t - (R/L)i' - (1/LC)i \]

\[ X'_s = V_m \]
\[ V'_m = F - (kX_s + bV_m)/m \]
\[ X''_s = (F_0 / m) \sin \omega_0 t - (b/m) X'_s - (k/m) X_s \]
Sinusoidal input

\[ i'' + \frac{R}{L}i' + \frac{1}{LC}i = \left(\omega_0 V_0 / L\right) \cos \omega_0 t \]

Assuming: \( i = A \sin(\omega_0 t) + B \cos(\omega_0 t) \)

\[ i' = A\omega_0 \cos(\omega_0 t) - B\omega_0 \sin(\omega_0 t) \]

\[ i'' = -A\omega_0^2 \sin(\omega_0 t) - B\omega_0^2 \cos(\omega_0 t) \]

\[ B = \frac{1-RA}{1/L\omega_0} \]

\[ A = \left[ R + \frac{1}{R} \left( \frac{1}{\omega_0 C} - L\omega_0 \right) \right]^{-1} \]
When $L\omega_0 = 1/\omega_0 C$, or $\omega_0 = \sqrt{1/LC}$

- $i = \max$
- $V_R = iR = \max$

Resonance occurs when the excitation frequency = the **natural frequency** of the system.
Forced vs Unforced Oscillations

At what frequencies can the systems shown oscillate?

(A) Both systems oscillate at $\omega^2 = 1/(LC)$
(B) Both systems oscillate at $\omega = \omega_0$
(C) Neither system oscillates
(D) System 1 oscillates at $\omega^2 = 1/(LC)$, but System 2 oscillates at $\omega = \omega_0$
(E) Impossible to tell
Forced vs Unforced Oscillations: with R

How do these systems differ?

(A) System 1 oscillates at approximately \( \omega^2 = 1/(LC) \)
(B) System 1 could have decaying oscillations
(C) System 1 could decay without oscillations
(D) System 1 could oscillate at constant amplitude forever
(E) System 2 could have decaying oscillations
(F) System 2 could oscillate at constant amplitude forever

\[ V = V_0 \sin \omega_0 t \]
Unforced Oscillations with Damping

\[ X_s'' + (b/m) X_s' + (k/m) X_s = 0 \]

\[ \begin{align*}
X &= C \exp(rt) \\
\frac{r^2 + (b/m)r + k/m}{2} &= 0
\end{align*} \]

\[ r = \frac{-b \pm \sqrt{(b)^2 - 4k/m}}{2m} \]

\[ i'' + (R/L) i' + (1/LC) i = 0 \]

\[ \begin{align*}
i &= C \exp(rt) \\
r^2 + (R/L)r + 1/LC &= 0
\end{align*} \]

\[ r = \frac{-R \pm \sqrt{(R)^2 - 4/LC}}{2L} \]

If \((R/L)^2 < 4/LC\), \(r\) will have imaginary components. There will be some oscillatory motion.

\[ r = \frac{1}{2} \left( -\frac{R}{L} \pm \sqrt{\frac{4}{LC} - \left(\frac{R}{L}\right)^2} \right) = -\frac{R}{2L} \pm \frac{1}{2} \sqrt{\frac{4}{LC} - \left(\frac{R}{L}\right)^2} j = -r_r \pm r_j
\]

\[ i = \exp(-r_r t) [\cos(r_j t) + B\sin(r_j t)] \]

Motion decays !!
• The Tacoma Narrows bridge did not fall due to linear resonance.
• It was a results of self excitation of the torsional motion.

https://www.youtube.com/watch?v=nFzu6CNtqec
The LRC circuit is shown, with switch, S, initially open and capacitor initially charged.

If R=0, L=1, C=2, the freq of oscillation of the current, $\omega$, will be

If R=0.5, L=1, C=2, the freq of oscillation of the current, $\omega$, will be

If R=0.5, L=0, C=2, the current, $\omega$, will have no oscillations  T/F