1. (10 points)

1-1. Find the real part, imaginary part and complex conjugate of the following complex numbers.

   a) $j^8 - 4j^{10} + j^7$
   b) $\frac{1}{3 + 2j}$
   c) $e^{j\pi/2}$

1-2. Solve for real numbers $x$ and $y$ such that the equality holds

\[
\frac{x + 1 + j(y - 3)}{5 + 3j} = 1 + j
\]

1-3. Express the following complex numbers using exponential form. (Hint: trigonometric identity $\sin^2 y + \cos^2 y = 1$ may be useful.)

   a) $-1$
   b) $1 + j\sqrt{3}$
2. (15 points)

A) Which of the following proposed functions, \( y(t) \), are the solutions to the differential equation: \( y'' - y' - 6y = 6t + 1 \).

- (TRUE/FALSE) \( y(t) = -t \)
- (TRUE/FALSE) \( y(t) = 6t + 1 \)
- (TRUE/FALSE) \( y(t) = e^{6t} \)
- (TRUE/FALSE) \( y(t) = e^{-2t} \)
- (TRUE/FALSE) \( y(t) = -t + e^{3t} \)
- (TRUE/FALSE) \( y(t) = 6t + 1 + e^{-2t} \)
- (TRUE/FALSE) \( y(t) = t + 1/6 + e^{6t} \)
- (TRUE/FALSE) \( y(t) = -t + e^{3t} + e^{-2t} \)

B) A system is described by a differential equation:

\[
\begin{bmatrix}
  x'_1 \\
  x'_2
\end{bmatrix} = \begin{bmatrix}
  0 & 2 \\
  -10 & 4
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
\]

which can be written as \( X' = AX \), where \( X \) is the state vector and \( A \) is the coefficients matrix. I have found two different solutions to the differential equation already: \( X = Y_1(t) \) and \( X = Y_2(t) \) where

\[
Y_1(t) = \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = \begin{bmatrix}
  e^{2t} \cos(4t) \\
  e^{2t} (\cos(4t) - B \sin(4t))
\end{bmatrix}
\]

\[
Y_2(t) = \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = \begin{bmatrix}
  e^{2t} \sin(4t) \\
  e^{2t} (\sin(4t) - C \cos(4t))
\end{bmatrix}
\]

Plug \( Y_1(t) \) and \( Y_2(t) \) into the differential equations and show with a proper choice of \( B \) and \( C \), both are solutions to the equations.

C) If the initial conditions are

\[
X(0) = \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = \begin{bmatrix}
  0 \\
  1
\end{bmatrix}
\]

Please find the particular solution to the problem in terms of \( X(t) = aY_1(t) + bY_2(t) \), where \( a \) and \( b \) should be found such that the initial condition is met.
3. (20 points)

For a given differential equation, e.g., \( x'' + \omega^2 x = 0 \), the steps to find the general solution are:

1) Find one or more forms of solutions

\( e^{\text{rt}} \) you find \( e^{-j\omega t} \) and \( e^{j\omega t} \)

2) Pick a set consisting only independent solutions

\( e^{-j\omega t}, e^{j\omega t} \)

3) Construct the general solution by multiplying every independent solution in the set by an arbitrary constant and sum them up

\( x = A e^{-j\omega t} + B e^{j\omega t} \)

4) The particular solution is found by applying initial conditions to find values for the constants in 3).

3-1. Using the general solution 3), find values of \( A \) and \( B \) such that the real-valued solution of \( x = D \cos(\omega t) \) can be found. Find different values of \( A \) and \( B \) such that the real-valued solution of \( x = E \sin(\omega t) \) can be found. \( D \) and \( E \) are unknown, real constants. \( A \) and \( B \) can be complex.

3-2. Show that if only \( e^{-j\omega t} \) is included in the set in 2) (and thus in 3)), then real-valued solutions of \( x = D \cos(\omega t) \) and \( x = E \sin(\omega t) \) cannot be found. (Hint: the easiest proof is to assume that \( x = D \cos(\omega t) \) and then try to find the constant \( A \) from the general solution such that this is true. You will end up with a contradiction (or only the trivial solution).)

3-3. The solutions \( \sin(\omega t) \) and \( \cos(\omega t) \) also form an independent set of solutions. Using these as your set in 2), verify that the complex exponential solutions in 1) can be obtained by setting proper constants of this general solution.
4. (20 points)

The equation of motion for the spring mass system to the right is derived in the web-text and solved both analytically and numerically. In this problem you will compare the analytic solution with that from Euler's method and consider the impact of the initial conditions.

4-1. Write the state equations for this system and re-derive the analytic solution.

4-2. Let $k_1=1000\text{N/m}$ and $m_2=250\text{Kg}$. Let $x_1(0)=0.3\text{m}$ and $v_2(0)=0\text{m/s}$.

a) Set up a matlab program to calculate the solution for $v_2(t)$ and $x_1(t)$ using the analytic solution and Euler’s method. Note the natural frequency of the system. (Hint: if you use a function for the state equation, like fn.m in the webtext, be sure that file is in the same directory as your main code.)

b) Plot your results for a step size of $dt=0.1$. Plot the curves on the same axes and label appropriately. Be sure your plots extend for at least 2 complete oscillations of the system. Comment on the results.

c) Consider a different set of initial conditions. Let $x_1(0)=0.3\text{m}$ and $v_2(0)=0.8\text{m/s}$. Note that the spring is stretched the same amount as before, but now the mass has a nonzero initial velocity. Plot your results (either Euler’s or analytic) for both sets of initial conditions on the same graph.

  is the period of oscillation the same? why (not)?
  is the amplitude of the oscillation the same? why (not)?
5. (20 points)

5-1. Check appropriate box(es) for the following differential equations of variable x. Note t is time and varies.

<table>
<thead>
<tr>
<th>Linear</th>
<th>Homogeneous</th>
<th>Constant coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>x'' + sin(t)*x' + exp(t)*x = 3 [ ] [ ] [ ] [ ]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>x'' + 2x' + x^2 = 0 [ ] [ ] [ ] [ ]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>x'' + exp(t)*x = 0 [ ] [ ] [ ] [ ]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>x'' + 3x = 3 [ ] [ ] [ ] [ ]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5-2. Pick a non-linear equation and prove that you cannot superpose solutions. (Hint: use proof by contradiction.)

5-3. Pick a linear non-homogeneous equation and prove that you cannot superpose solutions. (Hint: use proof by contradiction.)

5-4. Pick a linear homogeneous equation and prove that you can superpose solutions.

6. (15 points)

Consider that a mechanical system can be described by the two coupled state equations shown.

x' = -6x + 25v

v' = -x

6-1. Put the state equations into second order form.

6-2. Find the real valued general solution for this system as a function of time.