HW 2 SOLN

1-1 (Step 0 \(\rightarrow\) \(1/2\) pt each)

For A

\[ f_1 = k_1 x_1 \]
\[ f_2 = k_2 x_2 \]
\[ f_3 = b_3 v_3 \]
\[ f_4 = b_4 v_4 \]

For B

\[ f_1 = f_2 + f_3 \]
\[ f_2 = f_2 \]
\[ f_3 = b_3 v_3 \]
\[ f_4 = b_4 v_4 \]

(Step 1 \(\rightarrow\) \(1/2\) pt each)

For A

CL

\[ V_2 = V_3 \]
\[ \dot{V}_2 + V_2 + V_4 = 0 \]

GC

\[ \frac{f_1}{k_1} = \frac{f_2}{k_2} \]
\[ \frac{f_3}{b_3} = \frac{f_4}{b_4} \]

FB

\[ f_1 = f_2 + f_3 \]
\[ f_2 = f_2 \]
\[ f_3 = b_3 v_3 \]
\[ f_4 = b_4 v_4 \]

For B

CL

\[ V_2 = V_3 \]
\[ \dot{V}_2 + V_2 + V_4 = V_5 \]

GC

\[ \frac{f_1}{k_1} = \frac{f_2}{k_2} \]
\[ \frac{f_3}{b_3} = \frac{f_4}{b_4} \]

FB

\[ f_1 = f_2 + f_3 \]
\[ f_2 = f_2 \]
\[ f_3 = b_3 v_3 \]
\[ f_4 = b_4 v_4 \]

(Step 2 \(\rightarrow\) 1 pt per state vbls, 2 pts per EOM)

For A

State vbls: \( x_1 \), \( x_2 \), \( V_2 \)

\[ x_1' = V_1 = -V_2 - V_4 = -V_3 - \frac{f_4}{b_4} \]
\[ = -\frac{f_3}{b_3} - \frac{f_4}{b_4} \]
\[ = -\frac{(f_3 + f_2)}{b_3} - \frac{f_4}{b_4} \]
\[ = -\frac{(k_1 x_1 - k_2 x_2)}{b_3} \]
\[ = \frac{k_1 x_1}{b_3} - \frac{k_2 x_2}{b_3} \]

\[ x_2' = V_2 = V_3 \]
\[ = \frac{f_3}{b_3} \]
\[ = \frac{f_1 - f_2}{b_3} \]

\[ x_2 = k_1 x_1 - k_2 x_2 \]

For B

State vbls: \( x_1 \), \( x_2 \), \( V_5 \)

\[ x_1' = V_1 = V_5 - V_2 - V_4 \]
\[ = V_5 - V_3 - \frac{f_4}{b_4} \]
\[ = V_5 - \frac{f_3 + f_2}{b_3} - \frac{f_1}{b_4} \]
\[ = V_5 - \frac{(f_1 - f_2)}{b_3} - \frac{f_1}{b_4} \]
\[ = V_5 - \frac{(k_1 x_1 - k_2 x_2)}{b_3} \]
\[ = \frac{k_1 x_1}{b_3} - \frac{k_2 x_2}{b_3} \]

\[ x_2' = V_2 = V_3 \]
\[ = \frac{f_3}{b_3} \]
\[ = \frac{f_1 - f_2}{b_3} \]

\[ x_2 = k_1 x_1 - k_2 x_2 \]

\[ V_5 = a_S = \frac{1}{m_5} f_4 \]
\[ = \frac{1}{m_5} f_1 \]
\[ = \frac{1}{m_5} k_1 x_1 \]

\[ V_5 = \frac{-k_1 x_1}{m_5} \]
HW 2 SOLN

I. 2

- (1/2 pt for reason)
- (1/2 pt for reason)

- C

- D

- E

- Not needed - rigid mass is anchored to rigid wall
- Not needed - damper in parallel with rigid mass, means damper can't lengthen or shorten
- Not needed - damper not connected on RHS, so no velocity of v4

II. 3

- (1 pt each)
- OK if elements are numbered differently

- C

- D

- E

See next page for associated FBD's, GC's, FB's, CL's

- (a) T
- (b) T
- (c) T
- (d) T
- (e) T
- (f) F
- (g) F
- (h) T
- (i) T
- (j) F
- (k) \[ V_5 = V_2 \]
- (l) \[ x_5 > 0 \Rightarrow f_c > 0 \Rightarrow f_7 > 0 \Rightarrow v_7 > 0 \ (V_5 + V_c + V_7 = 0 \text{ from GC's}) \]
- (m) \[ F \Rightarrow m_2 \text{ can move right but accelerate left} \]
- (n) \[ T \Rightarrow a_4 < 0 \Rightarrow f_3 > 0 \Rightarrow x_3 > 0 \]
- (o) F \[ \text{not enough info} \]
For use with answering Prob #2

\[ V_1 = V_5 = V_2 \]
\[ V_3 = V_4 - V_2 \]
\[ V_5 + V_6 + V_7 = 0 \]

**GC's**

**FB**

\[ m_2a_2 = -f_1 - f_5 + f_3 + f_6 \]
\[ f_6 = f_7 \]
\[ m_4a_4 = -f_3 \]

**CL**

\[ f_1 = k_1 x_1 \]
\[ f_3 = k_3 x_3 \]
\[ f_5 = b_5 v_5 \]
\[ f_6 = k_6 x_6 \]
\[ f_7 = b_7 v_7 \]

3-1

DEQ: \( 3x' - 18x = 0 \) \( \Rightarrow \) \( x' - 6x = 0 \) \( \Rightarrow \) \( x' = 6x \)

Try \( x = Ae^{Bt} \) \( \Rightarrow \) plug into DEQ: \( AB e^{Bt} = 6A e^{Bt} \)

\( B = 6 \)

\( \Rightarrow \) \( x = Ae^{6t} \) ... general sol'n

Thus, (a) NO
(b) NO
(c) YES
(d) NO
(e) YES
(f) NO

1 pt each correct answer.

3-2

To find particular sol'n,
plug IC into general sol'n

\[ IC: x(0.1) = 10 = A e^{[6(0.1)]} \]
\[ 10 = A(1.822) \]
\[ A = 5.49 \]

Thus, \( x(t) = 5.49 e^{6t} \) \( \{4 \text{ pts}\} \)

or
\( x(t) = 5.5 e^{6t} \)
**Part 1**

**Step 1** (1 pt each)

- \[ f_B: -f_1 - f_2 = 0 \]
- \[ f_1 = f_2 \] or \[ f_1 = -f_2 \]

**CL**

- \[ f_1 = k_1 x_1 \]
- \[ f_2 = b_2 v_2 \]

**Step 2** (4 pts)

\[ x'_1 = v_1 = v_2 = \frac{f_2}{b_2} = -\frac{f_1}{b_2} = -\frac{k_1 x_1}{b_2} \]

\[ x_1(t) = \frac{-k_1}{b_2} x_1(t) \]

**Step 3** (6 pts) (See Matlab code and plots for numerical solution)

**Step 3 (Analytic)** (8 pts → 1 pt partial credit for each underlined part)

Try \( x_1 = A e^{bt} \) → plug into DEQ: \( A b e^{bt} = -\frac{k_1}{b_2} A e^{bt} \)

\[ B = \frac{-k_1}{b_2} = -1.5 \text{ s}^{-1} \]

General soln: \( x_1(t) = A e^{-1.5t} \)

**IC:** \( x_1(0) = 0.03 \)

\[ x_1(0) = 0.03 = Ae^0 \]

\[ A = 0.03 \]

\[ x_1(t) = 0.03 e^{-1.5t} \]

**Discussion:** In the numerical solution, when \( dt \) is smaller, it is closer to the exact (analytic) solution. This is because the error in the Euler Method solution decreases as \( dt \) is reduced (i.e., assumption of \( V_1 = \) const for small time steps is a better approximation for smaller \( dt \)).
clear;
k = 1500;
b = 1000;

%Time step size 1 (dt = 0.5)
dt1 = 0.5;
t1(1) = 0;
x1(1) = 0.03;
for i = 1:10
    v1(i) = (-k/b)*x1(i);
    x1(i+1) = x1(i) + v1(i)*dt1;
    t1(i+1) = t1(i) + dt1;
end

%Time step size 2 (dt = 0.1)
dt2 = 0.1;
t2(1) = 0;
x2(1) = 0.03;
for i = 1:50
    v2(i) = (-k/b)*x2(i);
    x2(i+1) = x2(i) + v2(i)*dt2;
    t2(i+1) = t2(i) + dt2;
end

texact = [0:0.01:5];
xexact = 0.03*exp(-1.5*texact);

plot(t1, x1, '-b', t2, x2, '--r', texact, xexact, '-k')
xlabel('time (s)')
ylabel('Spring displacement (m)')
legend('dt = 0.5s', 'dt = 0.1s', 'Analytic Solution')
Part 2

Step 1:  
1 pt each

FB:  \( m_3 a_3 = -f_1 - f_2 \)

G-C:  \( V_1 = V_2 = V_3 \)

CL:  \( f_1 = k_1 x_1 \)
\( f_2 = b_2 V_2 \)

Step 2: state vars: \( x_1, V_3 \)

2 pts each EOM

\[
\begin{align*}
x'_1 &= V_1 \\
x'_1 &= V_3 \\
V'_3 &= \frac{1}{m_3} (-f_1 - f_2) = \frac{1}{m_3} (-k_1 x_1 - b_2 V_2) \\
V'_3 &= \frac{-1}{m_3} (k_1 x_1 + b_2 V_3)
\end{align*}
\]

Step 3: Numerical Sol'n → See Matlab code & plots

Discussion:
First, the mass speeds up because the compressed spring pushes it to the right (+V direction). At later times (as the spring approaches its equilibrium), the damper (which is in tension as the mass moves) slows the mass down.
clear;
k = 2000;
b = 1200;
m = 100;

dt = 0.01;
t(1) = 0;
xs(1) = -0.02;
Vm(1) = 0;

for i = 1:300
    vs(i) = Vm(i);  % from spring EOM (xs')
    am(i) = -(1/m)*(k*xs(i) + b*Vm(i));  % from mass EOM (Vm')
    xs(i+1) = xs(i) + vs(i)*dt;  % Euler method for spring
    Vm(i+1) = Vm(i) + am(i)*dt;  % Euler method for mass
    t(i+1) = t(i) + dt;
end

subplot(2,1,1)
plot(t, xs)
xlabel('time (s)')
ylabel('Spring displacement (m)')

subplot(2,1,2)
plot(t, Vm)
xlabel('time (s)')
ylabel('Mass velocity (m/s)')